On the Relation Between Effort-Dominating and Symmetric Minitive Aggregation Operators

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Abstract. In this paper the recently introduced class of effort-dominating impact functions is examined. It turns out that each effort-dominating aggregation operator not only has a very intuitive interpretation, but also is symmetric minitive, and therefore may be expressed as a so-called quasi-I-statistic, which generalizes the well-known OWMin operator. These aggregation operators may be used e.g. in the Producer Assessment Problem which the most important instance is the scientometric/bibliometric issue of fair scientists' ranking by means of the number of citations received by their papers.

Keywords: Aggregation operators, impact functions, arity-monotonic, OWMax, OWMin, OMA, OWA, Hirsch's h-index, scientometrics.

Please cite this paper as:
Gagolewski M., *On the Relation Between Effort-Dominating and Symmetric Minitive Aggregation Operators*, In: Greco S. et al (Eds.), *Advances in Computational Intelligence*, Part III (*CCIS* **299**), Springer-Verlag, Heidelberg, 2012, pp. 276-285.

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1 Preliminaries

Information aggregation is a process that plays a very important role in many human activities, e.g. in statistics, engineering, and scientometrics. For example, in the Producers Assessment Problem [5,7] we are interested in the construction of a class of mappings that project the space of arbitrary-sized real vectors of individual goods' quality measures into a single number that reflects both (a) general quality of goods, and (b) the producer's overall productivity.

Nondecreasing, symmetric, and arity-monotonic aggregation operators useful in the PAP are called *impact functions*. For example, in [6] the most fundamental properties of L-, S-, quasi-L-, and quasi-S-statistics, which generalize OWA

[13], OWMax [2], OMA [10], and symmetric maxitive aggregation operators, respectively, were analyzed.

In [7] the class of effort-dominating operators was introduced. It was used to construct possibility distributions of impact functions' output values under — but not limited to — right-censored input data. As this very appealing class of aggregation operators has not been thoroughly examined yet, in this paper we are interested in finding how they are related to other functions known from the aggregation theory [cf. 8].

1.1 Notational Convention

From now on let $\mathbb{I} = [a, b]$ denote any closed interval of the extended real line, $\bar{\mathbb{R}} = [-\infty, \infty]$. The set of all arbitrary-length vectors with elements in $\mathbb{I}$, i.e. $\bigcup_{n=1}^{\infty} \mathbb{I}^n$, is denoted by $\mathbb{I}^{1,2,\dots}$. If not stated otherwise explicitly, we assume that $n, m \in \mathbb{N} = \{1, 2, \dots\}$. Moreover, let $[n] = \{1, 2, \dots, n\}$.

For any $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{I}^n$, we write $\mathbf{x} \le \mathbf{y}$ if and only if $(\forall i \in [n])\ x_i \le y_i$. Let $x_{(i)}$ denote the ith order statistic of $\mathbf{x} = (x_1, \dots, x_n)$. For $\mathbf{x}, \mathbf{y} \in \mathbb{I}^n$, we write $\mathbf{x} \cong \mathbf{y}$ if and only if there exists a permutation σ of the set $[n]$ such that $\mathbf{x} = (y_{\sigma(1)}, \dots, y_{\sigma(n)})$. A vector $(x, x, \dots, x) \in \mathbb{I}^n$ is denoted briefly by $(n * x)$. For each $\mathbf{x} \in \mathbb{I}^n$ and $\mathbf{y} \in \mathbb{I}^m$, $(\mathbf{x}, \mathbf{y})$ denotes the concatenation of the vectors, i.e. $(x_1, \dots, x_n, y_1, \dots, y_m) \in \mathbb{I}^{n+m}$.

If $\mathsf{f}, \mathsf{g} : \mathbb{I} \to \bar{\mathbb{R}}$ then $\mathsf{f} \preceq \mathsf{g}$ (g dominates f) if and only if $(\forall x \in \mathbb{I})\ \mathsf{f}(x) \le \mathsf{g}(x)$. The image of f is denoted by $\operatorname{img} \mathsf{f}$.

1.2 Aggregation Operators

Let $\mathcal{E}(\mathbb{I})$ denote the set of all **aggregation operators** in $\mathbb{I}^{1,2,\dots}$, i.e. $\mathcal{E}(\mathbb{I}) = \{\mathsf{F} : \mathbb{I}^{1,2,\dots} \to \mathbb{I}\}$. The class of aggregation operators reflects the very general idea of combining multiple numeric values into a single one, in some way representative of the whole input. Note that the aggregation (averaging) functions [cf. 8,11] form a particular subclass of aggregation operators.

In this paper we focus our attention on nondecreasing, arity-monotonic, and symmetric aggregation operators. Such operators are called **impact functions**[3].

Definition 1. *We say that* $\mathsf{F} \in \mathcal{E}(\mathbb{I})$ *is* ***nondecreasing****, denoted* $\mathsf{F} \in \mathcal{P}_{(\mathrm{nd})}$*, if*

$$(\forall n)\ (\forall \mathbf{x}, \mathbf{y} \in \mathbb{I}^n)\ \mathbf{x} \le \mathbf{y} \Longrightarrow \mathsf{F}(\mathbf{x}) \le \mathsf{F}(\mathbf{y}).$$

Definition 2. *We call* $\mathsf{F} \in \mathcal{E}(\mathbb{I})$ ***arity-monotonic****, denoted* $\mathsf{F} \in \mathcal{P}_{(\mathrm{am})}$*, if*

$$(\forall n, m)\ (\forall \mathbf{x} \in \mathbb{I}^n)\ (\forall \mathbf{y} \in \mathbb{I}^m)\quad \mathsf{F}(\mathbf{x}) \le \mathsf{F}(\mathbf{x}, \mathbf{y}).$$

[3] Originally, in [5,7] we have required impact functions to fulfill some additional boundary conditions, which are not needed in this context.

Definition 3. *We say that* $\mathsf{F} \in \mathcal{E}(\mathbb{I})$ *is* ***symmetric****, denoted* $\mathsf{F} \in \mathcal{P}_{(\mathrm{sym})}$*, if*

$$(\forall n)\ (\forall \mathbf{x}, \mathbf{y} \in \mathbb{I}^n)\ \mathbf{x} \cong \mathbf{y} \Longrightarrow \mathsf{F}(\mathbf{x}) = \mathsf{F}(\mathbf{y}).$$

Moreover, let us consider the following pre-order[4] on $\mathbb{I}^{1,2,\cdots}$. For any $\mathbf{x} \in \mathbb{I}^n$ and $\mathbf{y} \in \mathbb{I}^m$ we write $\mathbf{x} \trianglelefteq \mathbf{y}$ if and only if $n \le m$ and $x_{(n-i+1)} \le y_{(m-i+1)}$ for all $i \in [n]$. Recall that $x_{(n-i+1)}$ denotes the ith largest element of $\mathbf{x}$.

We have recently shown (see [7] for the proof) that an aggregation operator F satisfies the three above properties if and only if F is a morphism (order-preserving mapping) between the pre-ordered set $\left(\mathbb{I}^{1,2,\cdots}, \trianglelefteq\right)$ and $(\bar{\mathbb{R}}, \le)$.

Theorem 1. *Let* $\mathsf{F} \in \mathcal{E}(\mathbb{I})$*. Then* $\mathsf{F} \in \mathcal{P}_{(\mathrm{nd})} \cap \mathcal{P}_{(\mathrm{am})} \cap \mathcal{P}_{(\mathrm{sym})}$ *if and only if*

$$(\forall \mathbf{x}, \mathbf{y} \in \mathbb{I}^{1,2,\cdots})\ \mathbf{x} \trianglelefteq \mathbf{y} \Longrightarrow \mathsf{F}(\mathbf{x}) \le \mathsf{F}(\mathbf{y}). \tag{1}$$

2 Effort-Dominating Impact Functions

Given an aggregation operator $\mathsf{F} \in \mathcal{E}(\mathbb{I})$ and a constant $v \in \operatorname{img} \mathsf{F}$, let $\mathsf{F}^{-1}[v] := \left\{\mathbf{x} \in \mathbb{I}^{1,2,\cdots} : \mathsf{F}(\mathbf{x}) = v\right\}$ denote the *v-level set* of F. Additionally, if $\mathsf{F} \in \mathcal{P}_{(\mathrm{sym})}$ then, to avoid ambiguity, we assume that $\mathsf{F}^{-1}[v]$ consists only of vectors in $\mathbb{I}^{1,2,\cdots}$ that are unique w.r.t. the relation $\cong$ (e.g. their terms are sorted nonincreasingly).

Let us recall the notion of an effort-measurable aggregation operator [7].

Definition 4. *We say that* $\mathsf{F} \in \mathcal{P}_{(\mathrm{nd})} \cap \mathcal{P}_{(\mathrm{am})} \cap \mathcal{P}_{(\mathrm{sym})}$ *is* ***effort-measurable****, denoted* $\mathsf{F} \in \mathcal{P}_{(\mathrm{em})}$*, if* $\left(\mathsf{F}^{-1}[v], \trianglelefteq\right)$ *is a partially ordered set with a unique least element for any* $v \in \operatorname{img} \mathsf{F}$*.*

In other words, $\mathsf{F} \in \mathcal{P}_{(\mathrm{em})}$ if and only if for any $v \in \operatorname{img} \mathsf{F}$, $\left(\mathsf{F}^{-1}[v], \trianglelefteq\right)$ is a lower semilattice (a meet- or $\wedge$-semilattice).

Example 1. Not every $\mathsf{F} \in \mathcal{P}_{(\mathrm{nd})} \cap \mathcal{P}_{(\mathrm{am})} \cap \mathcal{P}_{(\mathrm{sym})}$ is effort-measurable. E.g. for a quasi-L-statistic [cf. 6] $\mathsf{L}_\triangle$ such that $\mathsf{L}_\triangle(x_1, \ldots, x_n) = \sum_{i=1}^n (n-i+1)\, x_{(n-i+1)}$, we have $\mathsf{L}_\triangle^{-1}[3] = \{(3), (1,1), (1.5, 0), (1,0,0), \ldots\}$, which has no least element w.r.t. $\trianglelefteq$. Moreover, the l_p-indices proposed in [4; cf. also 3] also are not effort-measurable. □

For any given $\mathsf{F} \in \mathcal{P}_{(\mathrm{em})}$ and $v \in \operatorname{img} \mathsf{F}$, let μ^v denote the least element of $\mathsf{F}^{-1}[v]$, i.e. $\mu^v := \min\{\mathsf{F}^{-1}[v]\}$. Clearly, for $w = \min\{\operatorname{img} \mathsf{F}\}$ we have $\mu^w = (a)$. Additionally, from now on $M(\mathsf{F}) := \{\mu^v : v \in \operatorname{img} \mathsf{F}\}$.

Example 2. Consider the aggregation operator $\mathsf{Max} \in \mathcal{P}_{(\mathrm{em})}$, defined as $\mathsf{Max}(x_1, \ldots, x_n) = x_{(n)}$ for $(x_1, \ldots, x_n) \in \mathbb{I}^{1,2,\cdots}$. We have $\operatorname{img} \mathsf{Max} = \mathbb{I}$, $\mathsf{Max}^{-1}[v] = \{(x_1, \ldots, x_n) \in \mathbb{I}^{1,2,\cdots} : x_{(n)} = v\}$, $\mu^v = (v) \in \mathbb{I}^1$, and $M(\mathsf{Max}) = \mathbb{I}^1$. □

[4] Formally, it is easily seen that $\trianglelefteq$ is not anti-symmetric (and hence is not a partial order, contrary to our statement in [7]) unless it is defined on the set of equivalence classes of $\cong$. Thanks to Prof. Michał Baczyński for pointing out this error.

From the class of effort-measurable aggregation operators let us distinguish the set of *effort-dominating* operators.

Definition 5. *We say that* $\mathsf{F} \in \mathcal{P}_{(\mathrm{em})}$ *is* ***effort-dominating****, denoted* $\mathsf{F} \in \mathcal{P}_{(\mathrm{ed})}$*, if* $(M(\mathsf{F}), \trianglelefteq)$ *is a chain.*

We see that in case of effort-dominating aggregation operators we have $\mu_v \triangleleft \mu_{v'} \iff v < v'$ for all $v, v' \in \operatorname{img} \mathsf{F}$. It is very important to note that each $\mathsf{F} \in \mathcal{P}_{(\mathrm{ed})}$ may be defined in the following, highly intuitive manner. For any $\mathbf{x} \in \mathbb{I}^{1,2,\cdots}$ it holds

$$\mathsf{F}(\mathbf{x}) = \operatorname*{arg\,max}_{v \in \operatorname{img} \mathsf{F}} \{\mu^v \in M(\mathsf{F}) : \mu^v \trianglelefteq \mathbf{x}\}. \tag{2}$$

We therefore look for the greatest v such that μ^v is still dominated by the input vector (cf. Fig. 1).

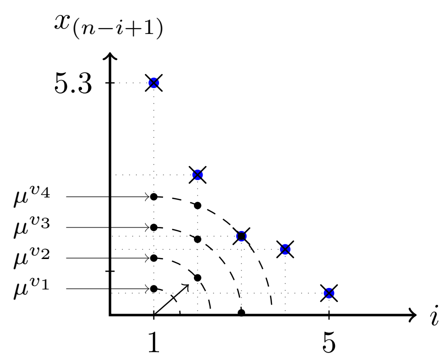

Fig. 1. $x_{(n-i+1)}$ as a function of i for $\mathbf{x} = (5.3, 3.2, 1.8, 1.5, 0.5)$ and the process of determining $\mathsf{F}(\mathbf{x})$.

Example 3. Let $\mathbb{I} = [0, \infty]$. The widely-known Hirsch's h index [9], introduced in 2005 in the field of bibliometrics, is an impact function H such that for $(x_1, \dots, x_n) \in \mathbb{I}^{1,2,\cdots}$ we have $\mathsf{H}(x_1, \dots, x_n) = \max\{i = 0, 1, \dots, n : x_{(n-i+1)} \geq i\}$ under the convention $x_{(n+1)} = x_{(n)}$. We have $\mu^0 = (0)$, and $\mu^n = (n * n)$ for $n \in \mathbb{N}$, therefore $\mathsf{H} \in \mathcal{P}_{(\mathrm{ed})}$. □

Example 4. Let $\mathbb{I} = [0, \infty]$. The r_p-index [3,4] for $p \geq 1$ is an impact function

$$\mathsf{r_p}(x_1, \dots, x_n) := \sup\{r \geq 0 : \mathbf{s}^{p,r} \trianglelefteq \mathbf{x}\},$$

where $(x_1, \dots, x_n) \in [0, \infty]^{1,2,\cdots}$ and $\mathbf{s}^{p,r} \in \mathbb{I}^{\lceil r \rceil}$, $r > 0$, denotes a sequence

$$\mathbf{s}^{p,r} = \begin{cases} \left(\sqrt[p]{r^p - 0^p}, \sqrt[p]{r^p - 1^p}, \dots, \sqrt[p]{r^p - \lceil r-1 \rceil^p}\right) & \text{if } p < \infty, \\ (r, r, \dots, r) & \text{if } p = \infty, \end{cases}$$

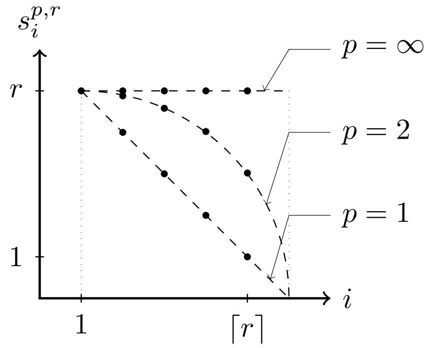

Fig. 2. $s_i^{p,r}$ as a function of i for $p = 1, 2, \infty$.

under the assumption $\mathbf{s}^{p,0} = (0)$, see Fig. 2.

It may be shown that for any $\mathbf{x} \in [0, \infty]^{1,2,\cdots}$, $\mathsf{r}_\infty(\lfloor \mathbf{x} \rfloor) = \mathsf{H}(\mathbf{x})$, and $\mathsf{r}_1(\lfloor \mathbf{x} \rfloor) = \mathsf{W}(\mathbf{x})$, where W is the Woeginger's w-index [12].

Clearly, each r_p-index is effort-dominating [7] — we have $\mu^v = \mathbf{s}^{v,r}$. The r_∞-index may be expressed as a symmetric maxitive or a symmetric modular aggregation operator [6]. However, for $x_{(2)} \ge x_{(1)} \ge 0$, e.g. we have $\mathsf{r}_1(x_{(2)}, x_{(1)}) = (x_{(2)} \wedge 2) \wedge (1 + (x_{(1)} \wedge 1))$, for which there do not exist nondecreasing functions $\mathsf{f}_{1,2}, \mathsf{f}_{2,2} : \mathbb{I} \to \bar{\mathbb{R}}$, such that $\mathsf{r}_1(x_{(2)}, x_{(1)}) = \mathsf{f}_{1,2}(x_{(2)}) \vee \mathsf{f}_{2,2}(x_{(1)})$, or $\mathsf{r}_1(x_{(2)}, x_{(1)}) = \mathsf{f}_{1,2}(x_{(2)}) + \mathsf{f}_{2,2}(x_{(1)})$.

□

3 Symmetric Minitive Aggregation Operators

Let us first recall the notion of a triangle of functions [cf. 6]:

Definition 6. *A* ***triangle of functions*** *is a sequence*

$$\triangle = (\mathsf{f}_{i,n})_{i \in [n], n \in \mathbb{N}},$$

where $(\forall n)$ $(\forall i \in [n])$ $\mathsf{f}_{i,n} : \mathbb{I} \to \mathbb{I}$.

Such objects may be used to generate interesting classes of aggregation operators, e.g. quasi-S- (consising of — but not limited to — all symmetric maxitive operators), and quasi-L-statistics (symmetric modular operators [cf. 10]). Here we introduce another one.

Definition 7. *A* ***quasi-I-statistic*** *generated by* $\triangle = (\mathsf{f}_{i,n})_{i \in [n], n \in \mathbb{N}}$ *is a function* $\mathsf{qI}_\triangle \in \mathcal{E}(\mathbb{I})$ *defined for any* $(x_1, \dots, x_n) \in \mathbb{I}^{1,2,\cdots}$ *as*

$$\mathsf{qI}_\triangle(\mathbf{x}) = \bigwedge_{i=1}^{n} \mathsf{f}_{i,n}(x_{(n-i+1)}). \tag{3}$$

Please note that $\wedge$ denotes the minimum (Infimum) operator, hence the name. We obviously have $\mathsf{qI}_{\triangle} \in \mathcal{P}_{(\mathrm{sym})}$ for any triangle of functions $\triangle$.

It is easily seen that quasi-I-statistics generalize OWMin operators [1], for which we have $\mathsf{f}_{i,n}(x) = c_{i,n} \vee x$ for some $c_{i,n} \in \mathbb{I}$, and symmetric minitive aggregation operators, i.e. the set of all $\mathsf{F} \in \mathcal{E}(\mathbb{I})$ such that $(\forall n)$ $(\forall \mathbf{x}, \mathbf{y} \in \mathbb{I}^n)$ it holds $\mathsf{F}(\mathbf{x} \overset{S}{\wedge} \mathbf{y}) = \mathsf{F}(\mathbf{x}) \wedge \mathsf{F}(\mathbf{y})$, where $\mathbf{x} \overset{S}{\wedge} \mathbf{y} = (x_{(n)} \wedge y_{(n)}, \ldots, x_{(1)} \wedge y_{(1)})$.

The following theorem states that, without loss of generality, triangles of functions generating nondecreasing quasi-I-statistics may be assumed to be of a particular form.

Theorem 2. *Let* $\mathbb{I} = [a, b]$ *and* $\triangle = (\mathsf{f}_{i,n})_{i \in [n], n \in \mathbb{N}}$. *Then* $\mathsf{qI}_{\triangle} \in \mathcal{P}_{(\mathrm{nd})}$ *if and only if there exists* $\triangledown = (\mathsf{g}_{i,n})_{i \in [n], n \in \mathbb{N}}$ *satisfying the following conditions:*

(i) $(\forall n)$ $(\forall i \in [n])$ $\mathsf{g}_{i,n}$ *is nondecreasing,*
(ii) $(\forall n)$ $(\forall i \in [n])$ $\mathsf{g}_{i,n}(b) = \mathsf{g}_{1,n}(b)$,
(iii) $(\forall n)$ $\mathsf{g}_{1,n} \preceq \cdots \preceq \mathsf{g}_{n,n}$,

such that $\mathsf{qI}_{\triangle} = \mathsf{qI}_{\triangledown}$.

Proof. $(\Longrightarrow)$ Let us fix n. Let $e_n = \mathsf{qS}_{\triangle}(n * b) = \bigwedge_{i=1}^{n} \mathsf{f}_{i,n}(b)$. Therefore, as $\mathsf{qI}_{\triangle} \in \mathcal{P}_{(\mathrm{nd})}$, for all $\mathbf{x} \in \mathbb{I}^n$ it holds $\mathsf{qI}_{\triangle}(\mathbf{x}) \leq e_n$. As a consequence,

$$\mathsf{qI}_{\triangle}(\mathbf{x}) = \bigwedge_{i=1}^{n} \mathsf{f}_{i,n}(x_{(n-i+1)}) = \bigwedge_{i=1}^{n} \left(\mathsf{f}_{i,n}(x_{(n-i+1)}) \wedge e_n\right).$$

Please note that, as $\mathsf{qI}_{\triangle}$ is nondecreasing, we have $(\forall \mathbf{x} \in \mathbb{I}^n)$ $(\forall i \in [n])$ $\mathsf{qI}_{\triangle}(\mathbf{x}) \leq \mathsf{qI}_{\triangle}((n-i) * b, i * x_{(i)})$, because $(x_{(n)}, \ldots, x_{(1)}) \leq ((n-i) * b, i * x_{(i)})$. We therefore have $\mathsf{qI}_{\triangle}(\mathbf{x}) \geq \mathsf{f}_{j,n}(x_{(n-i+1)})$, where $1 \leq i \leq j \leq n$. However, by definition, for each $\mathbf{x}$ there exists $k \in [n]$ for which $\mathsf{qI}_{\triangle}(\mathbf{x}) = \mathsf{f}_{k,n}(x_{(n-k+1)})$. Thus,

$$\begin{aligned}\mathsf{qI}_{\triangle}(\mathbf{x}) = {} & \mathsf{qI}_{\triangle}((n-1) * b, 1 * x_{(1)}) \\ & \wedge \mathsf{qI}_{\triangle}((n-2) * b, 2 * x_{(2)}) \\ & \vdots \\ & \wedge \mathsf{qI}_{\triangle}((n-n) * b, n * x_{(n)}).\end{aligned}$$

Consequently,

$$\mathsf{qI}_{\triangle}(\mathbf{x}) = \bigwedge_{i=1}^{n} \left(\bigwedge_{j=i}^{n} \mathsf{f}_{j,n}(x_{(n-i+1)}) \wedge e_n \right).$$

We may thus set $\mathsf{g}_{i,n}(x) := \bigwedge_{j=i}^{n} \mathsf{f}_{j,n}(x) \wedge e_n$ for all $i \in [n]$. We see that $\mathsf{g}_{1,n} \preceq \cdots \preceq \mathsf{g}_{n,n}$, and $\mathsf{g}_{1,n}(b) = \cdots = \mathsf{g}_{n,n}(b) = e_n$.

We will show that each $\mathsf{g}_{i,n}$ is nondecreasing. Assume otherwise. Let there exist i and $a \leq x < y \leq b$ such that $\mathsf{g}_{i,n}(x) > \mathsf{g}_{i,n}(y)$. We have $\mathsf{qS}_{\triangledown}((n-i) * b, i * x) = \mathsf{g}_{i,n}(x) > \mathsf{qS}_{\triangledown}((n-i) * b, i * y) = \mathsf{g}_{i,n}(y)$, a contradiction.

$(\Longleftarrow)$ Trivial. $\square$

Please note that, clearly, whenever $\triangledown$ fulfills all the above conditions then it holds $\operatorname{img} \mathsf{ql}_{\triangledown} = \bigcup_{n=1}^{\infty} \left(\bigcup_{i=1}^{n} \operatorname{img} \mathsf{g}_{i,n} \right)$.

Now let us find out when a nondecreasing quasi-I-statistic is arity-monotonic.

Theorem 3. *Let $\mathbb{I} = [a, b]$ and $\triangle = (\mathsf{f}_{i,n})_{i \in [n], n \in \mathbb{N}}$ be such that $(\forall n)$ $(\forall i \in [n])$ $\mathsf{f}_{i,n}$ is nondecreasing, $\mathsf{f}_{i,n}(b) = \mathsf{f}_{1,n}(b)$, and $\mathsf{f}_{1,n} \preceq \cdots \preceq \mathsf{f}_{n,n}$. Then $\mathsf{ql}_{\triangle} \in \mathcal{P}_{(\mathrm{am})}$ if and only if $(\forall n)$ $(\forall i \in [n])$ $\mathsf{f}_{i,n} \preceq \mathsf{f}_{i,n+1}$, and $\mathsf{f}_{n+1,n+1}(a) \geq \mathsf{f}_{1,n}(b)$.*

Proof. ($\Longrightarrow$) We have $(\forall x \in \mathbb{I})$ $\mathsf{ql}_{\triangle}(x) = \mathsf{f}_{1,1}(x)$. Moreover, $\mathsf{ql}_{\triangle}(x, a) = \mathsf{f}_{1,2}(x) \wedge \mathsf{f}_{2,2}(a)$. Therefore, $\mathsf{ql}_{\triangle}(x) \leq \mathsf{ql}_{\triangle}(x, a)$ if $\mathsf{f}_{2,2}(a) \geq \mathsf{f}_{1,1}(b)$ (when $x = b$) and $\mathsf{f}_{1,2} \succeq \mathsf{f}_{1,1}$.

Fix n. $\mathsf{ql}_{\triangle}(n * b) \leq \mathsf{ql}_{\triangle}(n * b, a)$ implies that $\mathsf{f}_{n+1,n+1}(a) \geq \mathsf{f}_{1,n}(b) = \cdots = \mathsf{f}_{n,n}(b)$. Now take arbitrary $\mathbf{x} \in \mathbb{I}^n$. $\mathsf{ql}_{\triangle}(x_{(n)}, (n-1) * a) \leq \mathsf{ql}_{\triangle}(x_{(n)}, (n) * a)$, implies that $\mathsf{f}_{1,n} \preceq \mathsf{f}_{1,n+1}$ (note that, according to the assumption, $\mathsf{f}_{1,n+1}(a) \leq \cdots \leq \mathsf{f}_{n+1,n+1}(a)$). For $\mathsf{ql}_{\triangle}(x_{(n)}, x_{(n-1)}, (n-2)*a) \leq \mathsf{ql}_{\triangle}(x_{(n)}, x_{(n-1)}, (n-1)*a)$ to hold for any $b \geq x_{(n)} \geq x_{(n-1)} \geq a$, we must have additionally $\mathsf{f}_{2,n} \preceq \mathsf{f}_{2,n+1}$. By considering the remaining $n-2$ terms of $\mathbf{x}$ we approach the condition $(\forall i \in [n])$ $\mathsf{f}_{i,n} \preceq \mathsf{f}_{i,n+1}$.

($\Longleftarrow$) Trivial. □

4 The Relationship Between the Two Classes

We are obviously interested in the relationship between the nondecreasing, arity-monotonic quasi-I-statistics and effort-dominating aggregation operators. It turns out that all effort-dominating aggregation operators belong to the class of quasi-I-statistics.

Theorem 4. *Let $\mathsf{F} \in \mathcal{P}_{(\mathrm{ed})}$ and $\mathbb{I} = [a, b]$. Then there exists $\triangle = (\mathsf{f}_{i,n})_{i \in [n], n \in \mathbb{N}}$ such that $\mathsf{ql}_{\triangle} = \mathsf{F}$.*

Proof. Take any $\mathsf{F} \in \mathcal{P}_{(\mathrm{ed})}$. For any n and $i \in [n]$, let $l_{i,n} := \min\{\mu^v_{(n-i+1)} : |\mu^v| = n\}$, and $u_{i,n} := \max\{\mu^v_{(n-i+1)} : |\mu^v| = n\}$. As $M(\mathsf{F})$ is a chain w.r.t. $\trianglelefteq$ and $\mathsf{F} \in \mathcal{P}_{(\mathrm{nd})} \cap \mathcal{P}_{(\mathrm{am})} \cap \mathcal{P}_{(\mathrm{sym})}$, we have $l_{i,n} \leq u_{i,n} \leq l_{i,n+1}$.

Let us first consider restriction of F to $\mathbb{I}^1$. For any $x_{(1)} \in \mathbb{I}$ we have:

$$\begin{aligned} \mathsf{F}(x_{(1)}) &= \max\{v : \mu^v_{(1)} \leq x_{(1)}, |\mu^v| = 1\} \\ &:= \mathsf{f}_{1,1}(x_{(1)}). \end{aligned}$$

Note that $\mathsf{f}_{1,1}$ is nondecreasing.

Furthermore, for $b \geq x_{(2)} \geq x_{(1)} \geq a$ it holds:

$$\mathsf{F}(x_{(2)}, x_{(1)}) = \begin{cases} \big(\max\{v : \mu^v_{(2)} \leq x_{(2)}, |\mu^v| = 2\} & \text{for} \quad x_{(2)} \geq l_{1,2} \\ \wedge \ \max\{v : \mu^v_{(1)} \leq x_{(1)}, |\mu^v| = 2\}\big) & \quad \text{and } x_{(1)} \geq l_{2,2}, \\ \mathsf{f}_{1,1}(x_{(2)}) & \text{otherwise,} \end{cases}$$

which may be written as:

$$\mathsf{F}(x_{(2)}, x_{(1)}) = \mathsf{f}_{1,2}(x_{(2)}) \wedge \mathsf{f}_{2,2}(x_{(1)}),$$

where:

$$\mathsf{f}_{1,2}(x_{(2)}) = \begin{cases} \mathsf{f}_{1,1}(x_{(2)}) & \text{for } x_{(2)} < l_{1,2}, \\ \max\{v : \mu^v_{(2)} \le x_{(2)}, |\mu^v| = 2\} & \text{otherwise,} \end{cases}$$

$$\mathsf{f}_{2,2}(x_{(1)}) = \begin{cases} \mathsf{f}_{1,1}(u_{1,1}) & \text{for } x_{(1)} < l_{2,2}, \\ \max\{v : \mu^v_{(1)} \le x_{(1)}, |\mu^v| = 2\} & \text{otherwise.} \end{cases}$$

Note that both $\mathsf{f}_{1,2}$ and $\mathsf{f}_{2,2}$ are nondecreasing, $\mathsf{f}_{1,1} \preceq \mathsf{f}_{1,2} \preceq \mathsf{f}_{2,2}$, $\mathsf{f}_{1,2}(b) = \mathsf{f}_{2,2}(b)$, and $\mathsf{f}_{2,2}(a) \ge \mathsf{f}_{1,1}(b) = \mathsf{f}_{1,1}(u_{1,1})$.

Now for $n = 3$, let $b \ge x_{(3)} \ge x_{(2)} \ge x_{(1)} \ge a$. It holds:

$$\mathsf{F}(x_{(3)}, x_{(2)}, x_{(1)}) = \begin{cases} \big(\max\{v : \mu^v_{(3)} \le x_{(3)}, |\mu^v| = 3\} & \text{for } \quad x_{(3)} \ge l_{1,3} \\ \wedge\ \max\{v : \mu^v_{(2)} \le x_{(2)}, |\mu^v| = 3\}\big) & \quad \text{and } x_{(2)} \ge l_{2,3}, \\ \wedge\ \max\{v : \mu^v_{(1)} \le x_{(1)}, |\mu^v| = 3\}\big) & \quad \text{and } x_{(1)} \ge l_{3,3}, \\ \mathsf{f}_{1,2}(x_{(3)}) \wedge \mathsf{f}_{2,2}(x_{(2)}) & \text{otherwise,} \end{cases}$$

which is equivalent to:

$$\mathsf{F}(x_{(3)}, x_{(2)}, x_{(1)}) = \mathsf{f}_{1,3}(x_{(3)}) \wedge \mathsf{f}_{2,3}(x_{(2)}) \wedge \mathsf{f}_{3,3}(x_{(1)}),$$

where

$$\mathsf{f}_{1,3}(x_{(3)}) = \begin{cases} \mathsf{f}_{1,2}(x_{(3)}) & \text{for } x_{(3)} < l_{1,3}, \\ \max\{v : \mu^v_{(3)} \le x_{(3)}, |\mu^v| = 3\} & \text{otherwise,} \end{cases}$$

$$\mathsf{f}_{2,3}(x_{(2)}) = \begin{cases} \mathsf{f}_{2,2}(x_{(2)}) & \text{for } x_{(2)} < l_{2,3}, \\ \max\{v : \mu^v_{(2)} \le x_{(2)}, |\mu^v| = 3\} & \text{otherwise,} \end{cases}$$

$$\mathsf{f}_{3,3}(x_{(1)}) = \begin{cases} \mathsf{f}_{1,2}(u_{1,2}) & \text{for } x_{(1)} < l_{3,3}, \\ \max\{v : \mu^v_{(1)} \le x_{(1)}, |\mu^v| = 3\} & \text{otherwise.} \end{cases}$$

By applying similar reasoning for any $n > 3$, we approach the equation

$$\mathsf{F}(x_{(n)}, \ldots, x_{(1)}) = \bigwedge_{i=1}^{n} \mathsf{f}_{i,n}(x_{(n-i+1)}),$$

where for $i < n$ we have

$$\mathsf{f}_{i,n}(x_{(n-i+1)}) = \begin{cases} \mathsf{f}_{i,n-1}(x_{(n-i+1)}) & \text{for } x_{(n-i+1)} < l_{i,n}, \\ \max\{v : \mu^v_{(n-i+1)} \le x_{(n-i+1)}, |\mu^v| = n\} & \text{otherwise,} \end{cases}$$

and

$$\mathsf{f}_{n,n}(x_{(1)}) = \begin{cases} \mathsf{f}_{1,n-1}(u_{1,n-1}) & \text{for } x_{(1)} < l_{n,n}, \\ \max\{v : \mu^v_{(1)} \le x_{(1)}, |\mu^v| = n\} & \text{otherwise.} \end{cases}$$

This formula generates each $\mathsf{f}_{i,n}$ being nondecreasing, $\mathsf{f}_{1,n} \preceq \cdots \preceq \mathsf{f}_{n,n}$, $\mathsf{f}_{i,n-1} \preceq \mathsf{f}_{i,n}$, $\mathsf{f}_{i,n}(b) = \mathsf{f}_{1,n}(b)$, and $\mathsf{f}_{n,n}(a) \geq \mathsf{f}_{1,n-1}(b)$, thus F is a nondecreasing, arity-monotonic quasi-I-statistic, which completes the proof. □

Example 5. By the construction above, we have:

$$\mathsf{r}_p(x_1,\ldots,x_n) = \begin{cases} \bigwedge_{i=1}^{n} \left((n \wedge x_{(n-i+1)}) \vee (i-1)\right) & \text{if } p = \infty, \\ \bigwedge_{i=1}^{n} \left(\left(n \wedge \sqrt[p]{x_{(n-i+1)}^p + (i-1)^p}\right) \vee (i-1)\right) & \text{if } p \in [1,\infty). \end{cases}$$

□

Interestingly, if we are given an already nondecreasingly-sorted input vector $\mathbf{x} \in \mathbb{I}^n$ and an effort-dominating impact function F for which the value $\max\{v : \mu^v_{(n-i+1)} \leq x_{(n-i+1)}, |\mu^v| = n\}$ may be computed in $O(1)$-time for all i,n (just as in Example 5) then $\mathsf{F}(\mathbf{x})$ may be computed in $O(n)$-time.

On the other hand, not each $\mathsf{ql}_\triangle \in \mathcal{P}_{(\mathrm{nd})} \cap \mathcal{P}_{(\mathrm{am})}$ belongs to $\mathcal{P}_{(\mathrm{ed})}$.

Example 6. Let $\mathbb{I} = [0,2]$ and $\triangle = (\mathsf{f}_{i,n})_{i\in[n],n\in\mathbb{N}}$ be such that $\mathsf{f}_{1,1} = \lfloor x \rfloor$, $\mathsf{f}_{1,2}(x) = x$, $\mathsf{f}_{2,2}(x) = 2$, and $(\forall n \geq 3)$ $(\forall i \in [n])$ $\mathsf{f}_{i,n}(x) = 2$. We see that $\mathsf{ql}_\triangle \in \mathcal{P}_{(\mathrm{em})} \cap \mathcal{P}_{(\mathrm{nd})} \cap \mathcal{P}_{(\mathrm{am})}$. However, e.g. $\mu^0 = (0) \trianglelefteq \mu^{0.5} = (0.5, 0) \ntrianglelefteq \mu^1 = (1)$. Therefore, $M(\mathsf{ql}_\triangle)$ is not a chain w.r.t. $\trianglelefteq$, and hence $\mathsf{ql}_\triangle \notin \mathcal{P}_{(\mathrm{ed})}$. □

For the sake of completeness, we shall show which triangles of functions generate effort-dominating quasi-I-statistics.

Theorem 5. *Let $\mathbb{I} = [a,b]$ and $\triangle = (\mathsf{f}_{i,n})_{i\in[n],n\in\mathbb{N}}$ be such that $(\forall n)$ $(\forall i \in [n])$ $\mathsf{f}_{i,n}$ is nondecreasing, $\mathsf{f}_{i,n}(b) = \mathsf{f}_{1,n}(b)$, $\mathsf{f}_{1,n} \preceq \cdots \preceq \mathsf{f}_{n,n}$, and $\mathsf{ql}_\triangle \in \mathcal{P}_{(\mathrm{am})}$. Then $\mathsf{ql}_\triangle \in \mathcal{P}_{(\mathrm{ed})}$ if and only if*

$$(\forall n)\ (\forall i \in [n])\ (\forall x < u_{i,n})\ \mathsf{f}_{i,n+1}(x) = \mathsf{f}_{i,n}(x),$$

where $u_{i,n} = \min\{y : \mathsf{f}_{i,n}(y) = \mathsf{f}_{i,n}(b)\}$.

Proof. ($\Longrightarrow$) Assume the opposite. Take the smallest n and the smallest $i \in [n]$ for which there exists $x < u_{i,n}$ such that $\mathsf{f}_{i,n}(x) < \mathsf{f}_{i,n+1}(x)$. We have 2 cases.

(i) Let $(\exists y)$ such that $\mathsf{f}_{i,n}(y) = \mathsf{f}_{i,n+1}(x)$. As each function is nondecreasing, $y > x$ holds. But in this case the least element (w.r.t. $\trianglelefteq$) of $\{\mathbf{x} : \mathsf{ql}_\triangle(\mathbf{x}) = \mathsf{f}_{i,n+1}(x)\}$ does not exist and hence $\mathsf{ql}_\triangle$ is not even effort-measurable.

(ii) Otherwise, we have $\mu^{\mathsf{f}_{i,n}(x)} \ntrianglelefteq \mu^{\mathsf{f}_{i,n+1}(x)}$ (note that $\mathsf{f}_{i,n}(x), \mathsf{f}_{i,n+1}(x) \in \operatorname{img} \mathsf{ql}_\triangle$), and therefore $M(\mathsf{ql}_\triangle)$ is not a chain, a contradiction.

($\Longleftarrow$) By arity-monotonicity, we have $(\forall n)$ $(\forall i \in [n])$ $\mathsf{f}_{i,n} \preceq \mathsf{f}_{i,n+1}$, and $\mathsf{f}_{n+1,n+1}(a) \geq \mathsf{f}_{1,n}(b)$. Take any $v, v' \in \operatorname{img} \mathsf{ql}_\triangle$ such that $v < v'$. Let us take the smallest m such that $(\exists x)$ $\mathsf{f}_{1,m}(x) = v$ and the smallest m' such that $(\exists y)$ $\mathsf{f}_{1,m}(y) = v'$. Note that, by the assumptions taken, $(\forall n < m)$ $(\forall \mathbf{x} \in \mathbb{I}^n)$ $\mathsf{ql}_\triangle(\mathbf{x}) < v$, and the same holds for m' and v'. Additionally, we have $m \leq m'$.

It holds $\mu^v_{(m-i+1)} = \min\{x : \mathsf{f}_{i,m}(x) \geq v\}$ for $i \in [m]$, and $\mu^{v'}_{(m'-j+1)} = \min\{x : \mathsf{f}_{j,m'}(x) \geq v'\}$ for $j \in [m']$. But $(\forall i \in [m])$ $\mu^v_{(m-i+1)} \leq \mu^{v'}_{(m'-i+1)}$, because $\mathsf{f}_{i,m}(\mu^v_{(m-i+1)}) = \mathsf{f}_{i,m'}(\mu^v_{(m-i+1)})$ and each function is nondecreasing. Therefore, $\mu^v \trianglelefteq \mu^{v'}$, QED. □

5 Conclusions

In this paper we have shown that all effort-dominating aggregation operators, among which we may find Gagolewski-Grzegorzewski's r_p-indices [3,4], Hirsch's h-index $= \mathsf{r}_\infty(\lfloor \mathbf{x} \rfloor)$ [9], and Woeginger's w-index $= \mathsf{r}_1(\lfloor \mathbf{x} \rfloor)$ [12], are symmetric-minitive.

Effort-dominating aggregation operators, proposed in [7], have a very intuitive interpretation: their value may be determined by comparing an input vector with elements of a set of "minimal quality requirements" needed to reach a particular "quality level".

Such aggregation operators may be used e.g. in the Producer Assessment Problem [see 5,7], whose most important instance is the issue of fair ranking of scientists by means of the number of citations received by their papers.

Acknowledgments. The author would like to thank Prof. Michał Baczyński, Prof. Przemysław Grzegorzewski, and Prof. Radko Mesiar for stimulating discussion. Also, the author would like to express his gratitude to K. Fokow for proof-reading of the manuscript.

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Please cite this paper as:
Gagolewski M., *On the Relation Between Effort-Dominating and Symmetric Minitive Aggregation Operators*, In: Greco S. et al (Eds.), *Advances in Computational Intelligence*, Part III (*CCIS* **299**), Springer-Verlag, Heidelberg, 2012, pp. 276-285.
