OM3: ordered maxitive, minitive, and modular aggregation operators — Part I: Axiomatic analysis under arity-dependence

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Abstract Recently, a very interesting relation between symmetric minitive, maxitive, and modular aggregation operators has been shown. It turns out that the intersection between any pair of the mentioned classes is the same. This result introduces what we here propose to call the OM3 operators. In the first part of our contribution on the analysis of the OM3 operators we study some properties that may be useful when aggregating input vectors of varying lengths. In Part II we will perform a thorough simulation study of the impact of input vectors' calibration on the aggregation results.

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1 Introduction

The process called aggregation of quantitative (numeric) data is of great importance in many practical domains. e.g. in mathematical statistics, engineering, operational research, and quality control. For instance, in scientometrics we are often interested in assessing scholars via aggregation of the citations number of each of their articles, or by using some other measures of their quality, see e.g. [12]. On the other hand,

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in marketing we sometimes need to synthesize the sales of a product with multiple model-ranges, i.e. to aggregate the number of versions (assessment of the product diversification) and the number of units sold for each model-range (assessment of market penetration) [6, 10].

The two above-presented examples are similar: they both concern summarization of quantitative data sets of nonuniform sizes. In classical approach to aggregation, however, the number of input values is fixed, see [2, 16, 15, 14]. To take into account such domains of applications, we shall rather consider arity-dependent [9] aggregation operators, cf. also [4, 3, 13, 19, 21].

In a recent article [8] some desirable properties of aggregation operators were considered: maxitivity, minitivity [14], and modularity [20, 18]. This result introduces a very appealing class of functions which we call here the OM3 operators. The OM3 operators include i.a. the well-known h-index [17], order statistics, and OWMax/OWMin operators [5]. In this paper we explore these functions under arity-dependence.

The paper is organized as follows. In Sec. 2 we introduce the OM3 operators and recall their most fundamental properties. Then, in Sec. 3 we study some desirable arity-dependent properties which are of interest in many practical situations, including their insensitivity to addition of elements equal to 0 or $\mathsf{F}(\mathbf{x})$ to the input vector $\mathbf{x}$, and sensitivity to addition of elements strictly greater than $\mathsf{F}(\mathbf{x})$. Finally, Sec. 4 concludes the paper. Moreover, in the second part of our contribution we perform a simulation study of the effect of inputs' calibration on the ranking of vectors by means of OM3 operators.

2 The OM3 aggregation operators

From now on let $\mathbb{I} = [0,b]$ denote a closed interval of the extended real line, possibly with $b = \infty$. The set of all vectors of arbitrary length with elements in $\mathbb{I}$, i.e. $\bigcup_{n=1}^{\infty} \mathbb{I}^n$, is denoted by $\mathbb{I}^{1,2,\dots}$. Moreover, let $\mathscr{E}(\mathbb{I})$ denote the set of all *aggregation operators* (also called extended aggregation functions) in $\mathbb{I}^{1,2,\dots}$, i.e. $\mathscr{E}(\mathbb{I}) = \{\mathsf{F} : \mathbb{I}^{1,2,\dots} \to \mathbb{I}\}$.

We see that the notion of an aggregation operator is very general: the only restriction is that it is a function into $\mathbb{I}$. Let us then focus our attention on operators that are nondecreasing (in each variable) and, additionally, symmetric (i.e. which do not depend on the order of elements' presentation) [11, 9].

Definition 1. We say that $\mathsf{F} \in \mathscr{E}(\mathbb{I})$ is *symmetric*, denoted $\mathsf{F} \in \mathscr{P}_{(\mathrm{sym})}$, if

$$(\forall n \in \mathbb{N})\ (\forall \mathbf{x}, \mathbf{y} \in \mathbb{I}^n)\ \mathbf{x} \cong \mathbf{y} \Longrightarrow \mathsf{F}(\mathbf{x}) = \mathsf{F}(\mathbf{y}),$$

where $\mathbf{x} \cong \mathbf{y}$ if and only if there exists a permutation σ of $[n] := \{1, 2, \dots, n\}$ such that $\mathbf{x} = (y_{\sigma(1)}, \dots, y_{\sigma(n)})$

Definition 2. We say that $\mathsf{F} \in \mathscr{E}(\mathbb{I})$ is *nondecreasing*, denoted $\mathsf{F} \in \mathscr{P}_{(\mathrm{nd})}$, if

$$(\forall n \in \mathbb{N})\ (\forall \mathbf{x}, \mathbf{y} \in \mathbb{I}^n)\ \mathbf{x} \le \mathbf{y} \Longrightarrow \mathsf{F}(\mathbf{x}) \le \mathsf{F}(\mathbf{y}),$$

where $\mathbf{x} \le \mathbf{y}$ if and only if $(\forall i \in [n])\ x_i \le y_i$.

We see that for each $\mathsf{F} \in \mathscr{P}_{(\mathrm{nd})}$ it holds $0 \le \mathsf{F}(n*0) \le \mathsf{F}(\mathbf{x}) \le \mathsf{F}(n*b) \le b$ for all $\mathbf{x} \in \mathbb{I}^n$, where $(n*y)$, $y \in \mathbb{I}$, denotes a vector $(y, y, \dots, y) \in \mathbb{I}^n$.

Let us recall the notion of symmetrized maxitivity, minitivity, and modularity used in [8], cf. also [14, 20, 18]. For $\mathbf{x}, \mathbf{y} \in \mathbb{I}^n$ let $\mathbf{x} \overset{S}{\vee} \mathbf{y} = (x_{(1)} \vee y_{(1)}, \dots, x_{(n)} \vee y_{(n)})$ and $\mathbf{x} \overset{S}{\wedge} \mathbf{y} = (x_{(1)} \wedge y_{(1)}, \dots, x_{(n)} \wedge y_{(n)})$, where $x_{(i)}$ denotes the ith order statistic of $\mathbf{x} \in \mathbb{I}^n$, i.e. the i-th smallest value in $\mathbf{x}$.

Definition 3. Let $\mathsf{F} \in \mathscr{E}(\mathbb{I})$. Then we call F a *symmetric maxitive* aggregation operator (denoted $\mathsf{F} \in \mathscr{P}_{(\mathrm{smax})}$), whenever $(\forall n)\ (\forall \mathbf{x}, \mathbf{y} \in \mathbb{I}^n)\ \mathsf{F}(\mathbf{x} \overset{S}{\vee} \mathbf{y}) = \mathsf{F}(\mathbf{x}) \vee \mathsf{F}(\mathbf{y})$.

Definition 4. Let $\mathsf{F} \in \mathscr{E}(\mathbb{I})$. Then F is *symmetric minitive* (denoted $\mathsf{F} \in \mathscr{P}_{(\mathrm{smin})}$), if $(\forall n)\ (\forall \mathbf{x}, \mathbf{y} \in \mathbb{I}^n)\ \mathsf{F}(\mathbf{x} \overset{S}{\wedge} \mathbf{y}) = \mathsf{F}(\mathbf{x}) \wedge \mathsf{F}(\mathbf{y})$.

Definition 5. Let $\mathsf{F} \in \mathscr{E}(\mathbb{I})$. Then F is *symmetric modular* (denoted $\mathsf{F} \in \mathscr{P}_{(\mathrm{smod})}$) whenever $(\forall n)\ (\forall \mathbf{x}, \mathbf{y} \in \mathbb{I}^n)\ \mathsf{F}(\mathbf{x} \overset{S}{\vee} \mathbf{y}) + \mathsf{F}(\mathbf{x} \overset{S}{\wedge} \mathbf{y}) = \mathsf{F}(\mathbf{x}) + \mathsf{F}(\mathbf{y})$.

It may be easily shown that $\mathscr{P}_{(\mathrm{smax})}, \mathscr{P}_{(\mathrm{smin})}, \mathscr{P}_{(\mathrm{smod})} \subseteq \mathscr{P}_{(\mathrm{sym})} \cap \mathscr{P}_{(\mathrm{nd})}$. Moreover, each symmetric modular aggregation operator is also symmetric additive (i.e. $\mathsf{F}(\mathbf{x} \overset{S}{+} \mathbf{y}) = \mathsf{F}(\mathbf{x}) + \mathsf{F}(\mathbf{y})$, where $\mathbf{x} \overset{S}{+} \mathbf{y} = (x_{(1)} + y_{(1)}, \dots, x_{(n)} + y_{(n)})$), cf. [10, 14].

Let us introduce the following class of aggregation operators.

Definition 6. Given $\mathbf{w} = (\mathsf{w}_1, \mathsf{w}_2, \dots)$, $\mathsf{w}_i : \mathbb{I} \to \mathbb{I}$, and a triangle of coefficients $\triangle = (c_{i,n})_{i \in [n], n \in \mathbb{N}}$, $c_{i,n} \in \mathbb{I}$, for any $\mathbf{x} \in \mathbb{I}^n$, let

$$\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) = \bigvee_{i=1}^{n} \mathsf{w}_n(x_{(n-i+1)}) \wedge c_{i,n}.$$

We see that the above contains i.a. all order statistics (whenever $\mathsf{w}_n(x) = x$, and $c_{i,n} = 0$, $c_{j,n} = b$ for $i < k$, $j \ge k$, and some k), OWMax operators (for $\mathsf{w}_n(x) = x$), and the famous Hirsch h-index ($\mathsf{w}_n(x) = \lfloor x \rfloor$, $c_{i,n} = i$).

It turns out that in case of nondecreasingness, with no loss in generality, we may assume that such aggregation operators are defined by $\mathsf{w}_1, \mathsf{w}_2, \dots$ and $\triangle$ of a specific form.

Lemma 1 (*Reduction*). $\mathsf{M}_{\triangle,\mathbf{w}} \in \mathscr{P}_{(\mathrm{nd})}$ *if and only if there exist* $\mathbf{w}' = (\mathsf{w}'_1, \mathsf{w}'_2, \dots)$, $\mathsf{w}'_i : \mathbb{I} \to \mathbb{I}$, *and a triangle of coefficients* $\nabla = (c'_{i,n})_{i \in [n], n \in \mathbb{N}}$ *satisfying the following conditions:*

(i) $(\forall n)$ w'_n *is nondecreasing,*

(ii) $(\forall n)\ c'_{1,n} \leq c'_{2,n} \leq \cdots \leq c'_{n,n}$,
(iii) $(\forall n)\ 0 \leq \mathsf{w}'_n(0) \leq c'_{1,n}$,
(iv) $(\forall n)\ \mathsf{w}'_n(b) = c'_{n,n}$,

such that $\mathsf{M}_{\triangle,\mathbf{w}} = \mathsf{M}_{\triangledown,\mathbf{w}'}$.

Proof. ($\Longrightarrow$) Let us fix n. Let $x, y \in \mathbb{I}$ be such that $x \leq y$. By $\mathscr{P}_{(\mathrm{nd})}$, $\mathsf{M}_{\triangle,\mathbf{w}}(n * x) = \bigvee_{i=1}^n \mathsf{w}_n(x) \wedge c_{i,n} = \mathsf{w}_n(x) \wedge \bigvee_{i=1}^n c_{i,n} \leq \mathsf{w}_n(y) \wedge \bigvee_{i=1}^n c_{i,n} = \mathsf{M}_{\triangle,\mathbf{w}}(n * y)$, where $\bigvee_{i=1}^n c_{i,n}$ is constant. Thus, we may set $\mathsf{w}'_n(z) := \mathsf{w}_n(z) \wedge \bigvee_{i=1}^n c_{i,n}$ for all $z \in \mathbb{I}$. Necessarily, w'_n is nondecreasing. Moreover, we may set $c'_{i,n} := c_{i,n} \wedge \mathsf{w}_n(b)$ and hence $\mathsf{w}'_n(b) = \bigvee_{i=1}^n c'_{i,n}$. Please note that $\mathsf{M}_{\triangle,\mathbf{w}} = \mathsf{M}_{\triangledown,\mathbf{w}'}$, where $\triangledown = (c'_{i,n})_{i\in[n]}$.

Let $d_n := \mathsf{M}_{\triangle,\mathbf{w}'}(n * 0) = \bigvee_{i=1}^n \mathsf{w}'_n(0) \wedge c'_{i,n} \geq 0$. Therefore, as $\mathsf{M}_{\triangle,\mathbf{w}'} \in \mathscr{P}_{(\mathrm{nd})}$, for all $\mathbf{x} \in \mathbb{I}^n$ it holds $\mathsf{M}_{\triangle,\mathbf{w}'}(\mathbf{x}) \geq d_n \geq 0$. As a consequence,

$$\begin{aligned}\mathsf{M}_{\triangle,\mathbf{w}'}(\mathbf{x}) &= \bigvee_{i=1}^n \mathsf{w}'_n(x_{(n-i+1)}) \wedge c'_{i,n} = \Big(\bigvee_{i=1}^n \mathsf{w}'_n(x_{(n-i+1)}) \wedge c'_{i,n}\Big) \vee d_n = \\ &= \bigvee_{i=1}^n (\mathsf{w}'_n(x_{(n-i+1)} \vee d_n) \wedge (c'_{i,n} \vee d_n).\end{aligned}$$

Therefore, we may set $\mathsf{w}'_n(y) := \mathsf{w}'_n(y) \vee d_n$ for all $y \in \mathbb{I}$, $c'_{i,n} := c'_{i,n} \vee d_n$, still with $\mathsf{M}_{\triangle,\mathbf{w}} = \mathsf{M}_{\triangledown,\mathbf{w}'}$, where $\triangledown = (c'_{i,n})_{i\in[n]}$. Since $c'_{i,n} \geq d_n$ for all i, then $\mathsf{M}_{\triangledown,\mathbf{w}'}(n * 0) = \mathsf{w}'_n(0)$, hence, $\mathsf{w}'_n(0) \leq \bigwedge_{i=1}^n c'_{i,n}$.

Fix any $\mathbf{x} \in \mathbb{I}^n$. We have:

$$\begin{aligned}\mathsf{M}_{\triangle',\mathbf{w}'}(\mathbf{x}) = \mathsf{M}_{\triangle',\mathbf{w}'}(&x_{(n)} \vee x_{(n-1)} \vee \cdots \vee x_{(2)} \vee x_{(1)}, \\ &x_{(n-1)} \vee \cdots \vee x_{(2)} \vee x_{(1)}, \\ &\ldots, \\ &x_{(2)} \vee x_{(1)}, \\ &x_{(1)}).\end{aligned}$$

As w'_n is nondecreasing, we get $\mathsf{w}'_n(x_{(n)} \vee \cdots \vee x_{(1)}) = \mathsf{w}'_n(x_{(n)}) \vee \cdots \vee \mathsf{w}'_n(x_{(1)})$. This implies

$$\mathsf{M}_{\triangledown,\mathbf{w}'}(\mathbf{x}) = \bigvee_{i=1}^n \Big[(\mathsf{w}'_n(x_{(n-i+1)})) \wedge (\bigvee_{j=1}^i c'_{j,n})\Big].$$

Now we may set $c'_{i,n} := \bigvee_{j=1}^i c'_{j,n}$, and still $\mathsf{M}_{\triangle,\mathbf{w}} = \mathsf{M}_{\triangledown,\mathbf{w}'}$. It is clear to see that $\mathsf{w}'_n(0) \leq c'_{1,n} \leq \cdots \leq c'_{n,n} = \mathsf{w}'_n(b)$.

($\Longleftarrow$) Let us fix n. It suffices to show that if w'_n and $\triangledown = (c'_{i,n})_{i\in[n],n\in\mathbb{N}}$ fulfill conditions (i)–(iv) then $\mathsf{M}_{\triangledown,\mathbf{w}'}$ is nondecreasing. Let $\mathbf{x}, \mathbf{y} \in \mathbb{I}^n$ be such that $\mathbf{x} \leq \mathbf{y}$. It is clear to see that $x_{(n-i+1)} \leq y_{(n-i+1)}$ for all i. Since w'_n is nondecreasing, we have $\mathsf{w}'_n(x_{(n-i+1)}) \wedge c'_{i,n} \leq \mathsf{w}'_n(y_{(n-i+1)}) \wedge c'_{i,n}$. Thus, $\bigvee_{i=1}^n \mathsf{w}'_n(x_{(n-i+1)}) \wedge c'_{i,n} \leq \bigvee_{i=1}^n \mathsf{w}'_n(y_{(n-i+1)}) \wedge c'_{i,n}$, which completes the proof. □

Most importantly, we have the following result, see [8, Theorem 20].

Theorem 1. *Let* $\mathbf{w}$ *and* $\triangle$ *be of the form given in Lemma 1. Then for all* $\mathbf{x} \in \mathbb{I}^{1,2,\dots}$

$$\begin{aligned}
\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) &= \bigvee_{i=1}^{n} \mathsf{w}_n(x_{(n-i+1)}) \wedge c_{i,n} \\
&= \bigwedge_{i=1}^{n} (\mathsf{w}_n(x_{(n-i+1)}) \vee c_{i-1,n}) \wedge c_{n,n} \\
&= \sum_{i=1}^{n} \left(\left(\mathsf{w}_n(x_{(n-i+1)}) \vee c_{i-1,n} \right) \wedge c_{i,n} - c_{i-1,n} \right).
\end{aligned}$$

with convention $c_{0,n} = 0$*.*

We see that $\mathsf{M}_{\triangle,\mathbf{w}}$ are symmetric maxitive, minitive and modular. What is more, by [8, Theorem 19], these are the only aggregation operators that belong to $\mathscr{P}_{(\mathrm{smax})} \cap \mathscr{P}_{(\mathrm{smin})} = \mathscr{P}_{(\mathrm{smax})} \cap \mathscr{P}_{(\mathrm{smod})} = \mathscr{P}_{(\mathrm{smin})} \cap \mathscr{P}_{(\mathrm{smod})} = \mathscr{P}_{(\mathrm{smax})} \cap \mathscr{P}_{(\mathrm{smod})} \cap \mathscr{P}_{(\mathrm{smin})}$. This is the reason why from now on we propose to call all $\mathsf{M}_{\triangle,\mathbf{w}}$ the *OM3* operators, i.e. *ordered maxitive, minitive, and modular* aggregation operators.

3 Some arity-dependent properties

Note that up to now our discussion concerned a fixed sample size n. Here we consider some properties that take into account the behavior of the aggregation operator when a new element is added to the input vector. This situation often occurs in practice: a "producer" whose quality has to be assessed "outputs" yet another "product" and we have to reevaluate his/her rating.

3.1 Zero-insensitivity

For each $\mathbf{x} \in \mathbb{I}^n$ and $\mathbf{y} \in \mathbb{I}^m$, let $(\mathbf{x}, \mathbf{y})$ denote the concatenation of the two vectors, i.e. $(x_1, \dots, x_n, y_1, \dots, y_m) \in \mathbb{I}^{n+m}$. In some applications, it is desirable to guarantee that if we add an element with rating 0, then the valuation of the vector does not change. It is because 0 may denote a minimal quality measure needed for an item to be taken into account in the aggregation process (note, however, a very different approach e.g. in [3] where *averaging* is considered). Such a property is called zero-insensitivity, see [9] and also [22]. More formally:

Definition 7. We call $\mathsf{F} \in \mathscr{E}(\mathbb{I})$ a *zero-insensitive* aggregation operator, denoted $\mathsf{F} \in \mathscr{P}_{(\mathrm{a0})}$, if for each $\mathbf{x} \in \mathbb{I}^{1,2,\dots}$ it holds $\mathsf{F}(\mathbf{x}, 0) = \mathsf{F}(\mathbf{x})$.

In other words, if F is zero-insensitive, then 0 is its so-called extended neutral element, see [14, Def. 2.108]. What is more, 0 is an idempotent element of every zero-insensitive function F such that $\mathsf{F}(0) = 0$.

It is easily seen that $\mathscr{P}_{(\mathrm{a0})} \cap \mathscr{P}_{(\mathrm{nd})} \subseteq \mathscr{P}_{(\mathrm{am})} \cap \mathscr{P}_{(\mathrm{nd})}$, where $\mathscr{P}_{(\mathrm{am})}$ denotes arity-monotonic aggregation operators, see [9], such that $(\forall \mathbf{x}, \mathbf{y} \in \mathbb{I}^{1,2,\dots})\ \mathsf{F}(\mathbf{x}) \leq \mathsf{F}(\mathbf{x}, \mathbf{y})$.

Concerning OM3 aggregation operators, we have what follows.

Theorem 2. *Let* $\mathbf{w}$ *and* $\triangle$ *be of the form given in Lemma 1. Then* $\mathsf{M}_{\triangle,\mathbf{w}} \in \mathscr{P}_{(\mathrm{a0})}$ *if and only if* $(\forall n)$ $(\forall i \in [n])$ $c_{i,n} = c_{i,n+1}$*, and*

(i) if x *s.t.* $\mathsf{w}_n(x) < c_{n,n}$*, then* $\mathsf{w}_n(x) = \mathsf{w}_{n+1}(x)$*,*
(ii) if x *s.t.* $\mathsf{w}_n(x) = c_{n,n}$*, then* $\mathsf{w}_{n+1}(x) \geq c_{n,n}$*.*

In other words, in such case $\mathsf{M}_{\triangle,\mathbf{w}} \in \mathscr{P}_{(\mathrm{a0})}$ if and only if there exists a nondecreasing function w and a sequence $(c_1, c_2, \dots)$ such that $(\forall n)$ $\mathsf{w}_n = \mathsf{w} \wedge c_n$ and

$$\triangle = \begin{pmatrix} c_{1,1} & & & & \\ c_{1,2} & c_{2,2} & & & \\ c_{1,3} & c_{2,3} & c_{3,3} & & \\ \vdots & \vdots & \vdots & \ddots & \\ \shortparallel & \shortparallel & \shortparallel & \dots & \dots \\ c_1 & c_2 & c_3 & \dots & \dots \end{pmatrix}$$

Proof. $(\Longrightarrow)$ Take any $\mathsf{M}_{\triangle,\mathbf{w}} \in \mathscr{P}_{(\mathrm{a0})}$ from Lemma 1. Let $n = 1$. Then $\mathsf{M}_{\triangle,\mathbf{w}}(x) = \mathsf{w}_1(x) \wedge c_{1,1}$ and $\mathsf{M}_{\triangle,\mathbf{w}}(x,0) = (\mathsf{w}_2(x) \wedge c_{1,2}) \vee (\mathsf{w}_2(0) \wedge c_{2,2})$ for any $x \in \mathbb{I}$. Please note that as $\mathsf{M}_{\triangle,\mathbf{w}}$ is nondecreasing, we have $\mathsf{w}_2(0) \leq c_{1,2} \leq c_{2,2}$ and w_2 is nondecreasing. Thus, $\mathsf{M}_{\triangle,\mathbf{w}}(x,0) = \mathsf{w}_2(x) \wedge c_{1,2}$. As $\mathsf{M}_{\triangle,\mathbf{w}} \in \mathscr{P}_{(\mathrm{a0})}$, it holds $\mathsf{M}_{\triangle,\mathbf{w}}(x) = \mathsf{M}_{\triangle,\mathbf{w}}(x,0)$, hence $\mathsf{w}_1(x) \wedge c_{1,1} = \mathsf{w}_2(x) \wedge c_{1,2}$. Let $x_1 = \inf\{x : \mathsf{w}_1(x) \geq c_{1,1}\}$. We shall consider two cases.

1. Let $x \leq x_1^-$.
   (a) If $\mathsf{w}_1(x) < c_{1,2}$, we have $\mathsf{w}_1(x) = \mathsf{w}_2(x)$.
   (b) If $\mathsf{w}_1(x) \geq c_{1,2}$, then $c_{1,1} > c_{1,2}$. Please note that $\mathsf{w}_1(x_1) \geq c_{1,1}$. Thus, $\mathsf{M}_{\triangle,\mathbf{w}}(x_1) = c_{1,1} = \mathsf{w}_2(x_1) \wedge c_{1,2} = \mathsf{M}_{\triangle,\mathbf{w}}(x_1, 0)$. Monotonicity of w_2 and case (a) implies $w_2(x_1) \geq c_{1,2}$. Thus, $c_{1,1} = c_{1,2}$, a contradiction.

   Hence, for all x such that $\mathsf{w}_1(x) < c_{1,1}$ we have $\mathsf{w}_1(x) = \mathsf{w}_2(x)$.
2. Now let us consider $x \geq x_1^+$. Please note that, as w_1 is nondecreasing and $\mathsf{w}_1(b) = c_{1,1}$, we have $\mathsf{w}_1(x) = c_{1,1}$. Thus, $c_{1,1} = \mathsf{w}_2(x) \wedge c_{1,2}$. From previous case and the fact that w_2 is nondecreasing, we have $\mathsf{w}_2(x) \geq c_{1,2}$. Hence, $c_{1,2} = c_{1,1}$.

Let $n = 2$. By $\mathscr{P}_{(\mathrm{a0})}$, we have $\mathsf{M}_{\triangle,\mathbf{w}}(x,0) = \mathsf{M}_{\triangle,\mathbf{w}}(x, 2*0)$. Thus, $\mathsf{w}_2(x) \wedge c_{1,2} = \mathsf{w}_3(x) \wedge c_{1,3}$. By similar steps as the above-performed, we get $c_{1,3} = c_{1,2} = c_{1,1}$ and $\mathsf{w}_3 = \mathsf{w}_2 = \mathsf{w}_1$ for $x < x_1$. For $\mathbf{x} = (x,x)$ we have $\mathsf{M}_{\triangle,\mathbf{w}}(x,x) = \mathsf{M}_{\triangle,\mathbf{w}}(2*x, 0) \Leftrightarrow \mathsf{w}_2(x) \wedge c_{2,2} = \mathsf{w}_3(x) \wedge c_{2,3}$. Likewise, we get $c_{2,2} = c_{2,3}$, $\mathsf{w}_2(x) = \mathsf{w}_3(x)$ for x such that $\mathsf{w}_2(x) < c_{2,2}$, and $\mathsf{w}_3(x) \geq c_{2,2}$ for x for which $\mathsf{w}_2(x) = c_{2,2}$.

The above reasoning may easily be extended for all other n.

$(\Longleftarrow)$ Please note that when conditions given in the right side of Theorem 2 hold, we may set $\mathsf{w}_n := \mathsf{w} \wedge c_{n,n}$ and $c_{i,n} := c_i$ for all n and some nondecreasing w and

c_i. This notion generates $\mathbf{w}$ and $\triangle$ fulfill conditions given in Lemma 1 such that $\mathsf{M}_{\triangle,\mathbf{w}} \in \mathscr{P}_{(\mathrm{nd})}$.

We will now show that $\mathsf{M}_{\triangle,\mathbf{w}} \in \mathscr{P}_{(\mathrm{a0})}$. Assume contrary. There exists $\mathbf{x} \in \mathbb{I}^{1,2,\dots}$ such that $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) \neq \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x},0)$. As w is nondecreasing and $\mathsf{w}(0) \le c_1$, we have $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x},0) = \left(\bigvee_{i=1}^{n}(\mathsf{w}(x_{(n-i+1)}) \wedge c_i\right) \vee (\mathsf{w}(0) \wedge c_{n+1}) = \bigvee_{i=1}^{n}\mathsf{w}(x_{(n-i+1)}) \wedge c_i) = \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x})$, a contradiction, and the proof is complete. $\square$

3.2 *F-insensitivity*

Zero-sensitivity may be strengthened as follows, cf. [9] and [23, Axiom A1].

Definition 8. $\mathsf{F} \in \mathscr{E}(\mathbb{I})$ is *F-insensitive*, denoted $\mathsf{F} \in \mathscr{P}_{(\mathrm{F0})}$, if

$$(\forall \mathbf{x} \in \mathbb{I}^{1,2,\dots})\ (\forall y \in \mathbb{I})\ y \le \mathsf{F}(\mathbf{x}) \Longrightarrow \mathsf{F}(\mathbf{x},y) = \mathsf{F}(\mathbf{x}).$$

Thus, we see that in this property we do not want to distinct a “producer” in any special way if he/she outputs a “product” with valuation not greater than his/her current overall rating.

Please note that $\mathscr{P}_{(\mathrm{F0})} \cap \mathscr{P}_{(\mathrm{nd})} \subseteq \mathscr{P}_{(\mathrm{a0})} \cap \mathscr{P}_{(\mathrm{nd})}$. Moreover, if $\mathsf{F} \in \mathscr{P}_{(\mathrm{a0})} \cap \mathscr{P}_{(\mathrm{nd})}$, then $\mathsf{F} \in \mathscr{P}_{(\mathrm{F0})}$ iff $(\forall \mathbf{x} \in \mathbb{I}^{1,2,\dots})\ \mathsf{F}(\mathbf{x},\mathsf{F}(\mathbf{x})) = \mathsf{F}(\mathbf{x})$. Note also that the property $\mathsf{F}(\mathbf{x},\mathsf{F}(\mathbf{x})) = \mathsf{F}(\mathbf{x})$, introduced in [24], is known as self-identity. A similar property, called *stability*, was also considered in [1].

Theorem 3. *Let* $\mathbf{w}$ *and* $\triangle$ *be of the form given in Lemma 1. Then* $\mathsf{M}_{\triangle,\mathbf{w}} \in \mathscr{P}_{(\mathrm{F0})}$ *if and only if there exists:*

(i) a nondecreasing function w*, for which if there exists* x *such that* $\mathsf{w}(x) > x$*, then* $(\forall y \in [x,\mathsf{w}(x)])\ \mathsf{w}(y) = \mathsf{w}(x)$*,*
(ii) a nondecreasing sequence $(c_1,c_2,\dots)$*, such that* $(\forall i)\ c_i \notin \{x \in \mathbb{I} : x < \mathsf{w}(x)\}$*,*

such that $\mathsf{w}_n = \mathsf{w} \wedge c_n$ *and* $c_{i,n} = c_i$*.*

Proof. $(\Longrightarrow)$ Let c_n be such that $c_n < \mathsf{w}_n(c_n)$ and $c_{n+1} > c_n$ for some n. Take $\mathbf{x} = (n * c_n)$, then $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) = \mathsf{w}_n(c_n) \wedge c_n = c_n$. Since $\mathsf{M}_{\triangle,\mathbf{w}} \in \mathscr{P}_{(\mathrm{F0})}$, we have $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) = \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x},c_n) = \mathsf{w}_n(c_n) \wedge c_{n+1}$, a contradiction, because $c_{n+1} > c_n$ and $\mathsf{w}_n(c_n) > c_n$. Thus, it is easily seen that for all i we have $\mathsf{w}(c_i) \le c_i$.

Let us now consider $y \in \mathbb{I}$ such that $\mathsf{w}(y) > y$. There is no loss in generality in assuming that $\mathsf{w}(y) \in (c_{n-1},c_n]$. $\mathsf{M}_{\triangle,\mathbf{w}}((n-1)*b,y) = (\mathsf{w}(b) \wedge c_{n-1}) \vee (\mathsf{w}(y) \wedge c_n) = c_{n-1} \vee \mathsf{w}(y) = \mathsf{w}(y)$. Moreover, $\mathsf{M}_{\triangle,\mathbf{w}}((n-1)*b,\mathsf{w}(y),y) = (\mathsf{w}(b) \wedge c_{n-1}) \vee (\mathsf{w}(\mathsf{w}(y)) \wedge c_n) \vee (\mathsf{w}(y) \wedge c_{n+1}) = (\mathsf{w}(\mathsf{w}(y)) \wedge c_n) \vee \mathsf{w}(y) = \mathsf{w}(\mathsf{w}(y)) \wedge c_n$. Since $\mathsf{M}_{\triangle,\mathbf{w}} \in \mathscr{P}_{(\mathrm{F0})}$, $\mathsf{M}_{\triangle,\mathbf{w}}((n-1)*b,y) = \mathsf{M}_{\triangle,\mathsf{w}}((n-1)*b,\mathsf{w}(y),y)$. Thus, $\mathsf{w}(y) = \mathsf{w}(\mathsf{w}(y)) \wedge c_n$. This implies that either $\mathsf{w}(y) = \mathsf{w}(\mathsf{w}(y))$ or $\mathsf{w}(y) = c_n$. Hence, $\mathsf{w}(y) \le c_n$. We shall consider the $\mathsf{w}(y) = \mathsf{w}(\mathsf{w}(y))$ case.

Let us take the largest interval $\mathbb{L}$, $y \in \mathbb{L}$, such that $(\forall x \in \mathbb{L})\ \mathsf{w}(x) > x$. Denote the bounds of this interval by x_1, x_2, respectively. $\mathbb{L}$ may be either left-open or left-closed depending on the kind of potential discontinuity of w at x_1, but this will not affect our reasoning here. Surely, however, it is right-open (by definition of $\mathbb{L}$ and nondecreasingness of w), we have $\mathsf{w}(x_2) = x_2$. Take any $y_1 \in \mathbb{L}$. From the previous paragraph, $y_1 < \mathsf{w}(y_1)$. Let $y_2 = \mathsf{w}(y_1)$. If $y_2 \notin \mathbb{L}$, then $\mathsf{w}(y_2) \leq y_2$ and from nondecreasingness of w we have that $y_2 = x_2$. On the other hand, let $y_2 \in \mathbb{L}$. By the definition of $\mathbb{L}$, we get $y_2 < \mathsf{w}(y_2)$ and, from the above- performed reasoning, as $\mathsf{M}_{\triangle,\mathbf{w}} \in \mathscr{P}_{(\mathrm{F0})} \cap \mathscr{P}_{(\mathrm{nd})}$, it follows $\mathsf{w}(y_2) = \mathsf{w}(\mathsf{w}(y_2))$. Let $y_3 = \mathsf{w}(y_2)$. Then $y_2 < y_3 = \mathsf{w}(y_3)$ and therefore $y_3 = \mathsf{w}(y_2) \notin \mathbb{L}$, a contradiction. Thus, $y_2 = x_2$ and $(\forall x \in \mathbb{L})\ \mathsf{w}(x) = \mathsf{w}(x_2)$.

($\Longleftarrow$) Assume otherwise. Thus, there exists $\mathbf{x} \in \mathbb{I}^{n-1}$ for some n such that $y := \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) \neq \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}, y)$. Moreover, let $\mathbf{x}' = (\mathbf{x}, 0)$ and $\mathbf{x}'' = (\mathbf{x}, y)$. By $\mathscr{P}_{(\mathrm{a0})}$, we have $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) = \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}')$. From definition, there exists k such that $y = \mathsf{w}(x'_{(n-k+1)}) \wedge c_k$. We shall now consider two cases.

1. If $y > x'_{(n-k+1)}$, then for some $l < k$ we have $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}'') = \mathsf{M}_{\triangle,\mathbf{w}}(x'_{(n)}, \ldots, x'_{(n-l+2)}, y, x'_{(n-l+1)}, \ldots, x'_{(n-k+1)}, \ldots, x'_{(2)}) = \left(\bigvee_{i=1}^{l-1} \mathsf{w}_n(x'_{(n-i+1)}) \wedge c_{i,n}\right) \vee (\mathsf{w}(y) \wedge c_{l,n}) \vee \left(\bigvee_{i=l+1}^{k} \mathsf{w}_n(x'_{(n-i+2)}) \wedge c_{i,n}\right) \vee \left(\bigvee_{i=k+1}^{n} \mathsf{w}_n(x'_{(n-i+2)}) \wedge c_{i,n}\right)$.
   (a) If $\mathsf{w}_n(x'_{(n-k+1)}) \leq c_{k,n}$, then $y = \mathsf{w}_n(x'_{(n-k+1)}) > x'_{(n-k+1)}$. Moreover, $\mathsf{w}_n(y) = \mathsf{w}_n(\mathsf{w}_n(x'_{(n-k+1)})) = \mathsf{w}_n(x'_{(n-k+1)})$. This implies $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}'') = \mathsf{w}_n(x'_{(n-k+1)}) \vee \left(\bigvee_{i=l+1}^{k} \mathsf{w}_n(x'_{(n-i+2)}) \wedge c_{i,n}\right) \neq \mathsf{w}_n(x'_{(n-k+1)}) = y$. But $\mathsf{w}_n(x'_{(n-i+2)})$ for $i = l+1, \ldots, k$ is equal to $\mathsf{w}_n(x'_{(n-k+1)})$, a contradiction.
   (b) If $\mathsf{w}_n(x'_{(n-k+1)}) > c_{k,n}$, then by (ii), $y = c_{k,n} \not> x'_{(n-k+1)}$.
2. Now assume that $y \leq x'_{(n-k+1)}$. Then for some $l \geq k$ we obviously have $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}'') = \mathsf{M}_{\triangle,\mathbf{w}}(x'_{(n)}, \ldots, x'_{(n-k+1)}, \ldots, x'_{(n-l+2)}, y, x'_{(n-l+1)}, \ldots, x'_{(2)})$ $= (\mathsf{w}_n(y) \wedge c_{l,n}) \vee \left(\bigvee_{i=1}^{l-1} \mathsf{w}_n(x'_{(n-i+1)}) \wedge c_{i,n}\right) \vee \left(\bigvee_{i=l+1}^{n} \mathsf{w}_n(x'_{(n-i+2)}) \wedge c_{i,n}\right) = (\mathsf{w}_n(x_{(n-k+1)}) \wedge c_{k,n}) \vee (\mathsf{w}_n(y) \wedge c_{l,n}) \vee \left(\bigvee_{i=l+1}^{n} \mathsf{w}_n(x'_{(n-i+2)}) \wedge c_{i,n}\right)$.
   (a) If $\mathsf{w}_n(x'_{(n-k+1)}) \leq c_{k,n}$, then for all $i > k$ we have $\mathsf{w}_n(x''_{(n-i+1)}) \leq \mathsf{w}_n(x'_{(n-k+1)}) = y \leq c_{k,n}$. Therefore, $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}'') = y$, a contradiction.
   (b) If $\mathsf{w}_n(x'_{(n-k+1)}) > c_{k,n}$, then by (ii), we have $\mathsf{w}_n(y) \leq c_{k,n}$. It implies that for all $i > l$ we have $\mathsf{w}_n(x_{(n-i+2)}) \leq \mathsf{w}_n(y) \leq c_{k,n}$. Thus, $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}'') = c_{k,n} = y$, a contradiction.

Hence, $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) = \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}, \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}))$ for any $\mathbf{x}$, and the proof is complete. □

3.3 *F+sensitivity*

Clearly, F-insensitivity does not guarantee that if a "producer" outputs an element with valuation greater than $\mathsf{F}(\mathbf{x})$, then his/her overall valuation is raised. As such situation may sometimes be desirable, let us then consider the property discussed in [9] and [23, Axiom A2].

Definition 9. $\mathsf{F} \in \mathscr{E}(\mathbb{I})$ is *F+sensitive*, denoted $\mathsf{F} \in \mathscr{P}_{(\mathrm{F+})}$, if

$$(\forall \mathbf{x} \in \mathbb{I}^{1,2,\dots})\ (\forall y \in \mathbb{I})\ y > \mathsf{F}(\mathbf{x}) \Longrightarrow \mathsf{F}(\mathbf{x}, y) > \mathsf{F}(\mathbf{x}).$$

Let us study when this property holds in case of OM3 operators. We will consider it under $\mathscr{P}_{(\mathrm{a0})}$, as otherwise the form of $\triangle$ and $\mathbf{w}$ gets very complicated and too inconvenient to be used in practical applications.

Theorem 4. *Let* $\mathbf{w}$ *and* $\triangle$ *be of the form given in Lemma 1. Then* $\mathsf{M}_{\triangle,\mathbf{w}} \in \mathscr{P}_{(\mathrm{a0})} \cap \mathscr{P}_{(\mathrm{F+})}$ *if and only if there exist:*

(i) a function w *such that* $\mathsf{w}(x) \geq x$ *for all* x*, and strictly increasing for* $x : \mathsf{w}(x) < \mathsf{w}(b)$*,*
(ii) a sequence $(c_1, c_2, \dots)$ *such that for* $c_i < \mathsf{w}(b)$ *we have* $\mathsf{w}(x) < c_i$ *for all* $x : \mathsf{w}(x) < \mathsf{w}(b)$ *and* $c_i < c_{i+1}$

such that $\mathsf{w}_n = \mathsf{w} \wedge c_n$ *and* $c_{i,n} = c_i$*.*

Before proceeding to the proof, note that if w is continuous at $x_d = \sup\{x \in \mathbb{I} : \mathsf{w}(x) < \mathsf{w}(b)\}$, then $c_1 = \mathsf{w}(b)$ and $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) = \mathsf{w}(x_{(n)})$.

Proof. ($\Longrightarrow$) First we will show that $\mathsf{w}(x) \geq x$ for all $x \in \mathbb{I}$. Assume otherwise. Take any x, $\mathsf{w}(x) < \mathsf{w}(b)$, such that $\mathsf{w}(x) < x$, and the smallest $c_k > \mathsf{w}(x)$. Therefore, $\mathsf{M}_{\triangle,\mathbf{w}}(k * x) = \mathsf{w}(x)$. Let us take $\varepsilon > 0$ such that $\mathsf{w}(x) \leq \mathsf{w}(x) + \varepsilon < x$. This implies $\mathsf{M}_{\triangle,\mathbf{w}}(k * x, \mathsf{w}(x) + \varepsilon) = (\mathsf{w}(x) \wedge c_k) \vee (\mathsf{w}(\mathsf{w}(x) + \varepsilon) \wedge c_{k+1}) = \mathsf{w}(x)$, a contradiction, since w is nondecreasing and $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) \in \mathscr{P}_{(\mathrm{F+})}$.

Take any $x \in \mathbb{I}$ such that $\mathsf{w}(x) < \mathsf{w}(b)$ and the smallest $c_k > \mathsf{w}(x)$. Then $\mathsf{M}_{\triangle,\mathbf{w}}(k * x) = \mathsf{w}(x)$. Let $\varepsilon > 0$. Then $\mathsf{w}(x) + \varepsilon > x$ and $\mathsf{M}_{\triangle,\mathbf{w}}(\mathsf{w}(x) + \varepsilon, k * x) = (\mathsf{w}(\mathsf{w}(x) + \varepsilon) \wedge c_1) \vee (\mathsf{w}(x) \wedge c_{k+1}) = (\mathsf{w}(\mathsf{w}(x) + \varepsilon) \wedge c_1) \vee \mathsf{w}(x) > \mathsf{w}(x)$. This implies $\mathsf{w}(\mathsf{w}(x) + \varepsilon) > \mathsf{w}(x)$. Thus, w must be strictly increasing. Moreover, $\mathsf{w}(x) < c_1$. We shall now consider two cases.

If w is continuous at x_d, then it is easily seen that $c_1 = \mathsf{w}(b)$ and $\mathsf{M}_{\triangle,\mathbf{w}} \in \mathscr{P}_{(\mathrm{F+})}$ for all $\mathbf{x}$ since w is nondecreasing and $\mathsf{w}(x) \geq x$.

If w is discontinuous at x_d, then for $x \in \mathbb{I}$ such that $\mathsf{w}(x) > c_1$ we have $\mathsf{w}(x) = \mathsf{w}(b)$. Therefore, $\mathsf{M}_{\triangle,\mathbf{w}}(x) = \mathsf{w}(x) \vee c_1 = c_1$. Let us take $\varepsilon > 0$. If $c_1 + \varepsilon < x$, then $\mathsf{M}_{\triangle,\mathbf{w}}(c_1 + \varepsilon, x) = (\mathsf{w}(c_1 + \varepsilon) \vee c_1) \wedge (\mathsf{w}(x) \wedge c_2) > c_1$. This implies $c_2 > c_1$. Otherwise $\mathsf{M}_{\triangle,\mathbf{w}}(x, c_1 + \varepsilon) = (\mathsf{w}(x) \vee c_1) \wedge (\mathsf{w}(c_1 + \varepsilon) \wedge c_2)$ and from $\mathscr{P}_{(\mathrm{F+})}$ we get $c_2 > c_1$. We continue in this fashion by considering vectors $(i * x)$ and we obtain $c_i < c_{i+1}$ for all i.

($\Longleftarrow$) As noted above, if w is continuous at x_d, then $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) = \mathsf{w}(x_{(n)})$. Thus, $\mathsf{M}_{\triangle,\mathbf{w}} \in \mathscr{P}_{(\mathrm{F+})}$ for all $\mathbf{x}$ since w is strictly increasing and $\mathsf{w}(x) \geq x$.

Let us now consider discontinuity at x_d. Assume that $\mathscr{P}_{(\mathrm{F+})}$ does not hold. Take $\mathbf{x} \in \mathbb{I}^{n-1}$ such that $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) < \mathsf{w}(b)$ and $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}') = \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}'')$, where $\mathbf{x}' = (\mathbf{x}, 0)$ and $\mathbf{x}'' = (\mathbf{x}, \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}') + \varepsilon)$ for some $\varepsilon > 0$. Please note that as $\mathsf{M}_{\triangle,\mathbf{w}} \in \mathscr{P}_{(\mathrm{a0})}$, we have $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) = \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}')$. If $\mathsf{w}(x'_{(n)}) < c_1$, then $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}') = \mathsf{w}(x'_{(n)}) < c_1$. Thus for $\varepsilon > 0$ we have $\mathsf{w}(x'_{(n)}) + \varepsilon > x'_{(n)}$ and either $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}'') = \mathsf{w}(\mathsf{w}(x'_{(n)}) + \varepsilon) > \mathsf{w}(x'_{(n)})$ if $\mathsf{w}(\mathsf{w}(x'_{(n)}) + \varepsilon) < c_1$ or $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}'') \geq c_1 > \mathsf{w}(x'_{(n)})$ if $\mathsf{w}(\mathsf{w}(x'_{(n)}) + \varepsilon) \geq c_1$. In both cases we have a contradiction. Therefore, if $(F+)$ does not hold, we surely have $\mathsf{w}(x'_{(n)}) \geq c_1$. This implies $\mathsf{w}(x'_{(n)}) = \mathsf{w}(b)$ and since $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}') < \mathsf{w}(b)$ and $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}') > c_1$, it must holds $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}') = c_k$ for some k. Therefore, $\mathsf{w}(x'_{(n-k+1)}) \geq c_1$ and $\mathsf{w}(x'_{(n-k)}) < c_1$. Take $\varepsilon > 0$. As $\mathsf{w}(c_k + \varepsilon) \geq c_k + \varepsilon > c_k > c_1$, we have $\mathsf{w}(c_k + \varepsilon) = \mathsf{w}(b)$. Hence, $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}'') = \bigvee_{i=1}^{k+1} (\mathsf{w}(x'_{(n-k+1)}) \wedge c_i) = c_{k+1} > c_k$, a contradiction. Thus, $\mathsf{M}_{\triangle,\mathbf{w}} \in \mathscr{P}_{(\mathrm{F}+)}$, QED. $\square$

4 Conclusions

In this paper we have considered a very interesting class of symmetric maxitive, minitive and modular aggregation operators, that is the OM3 operators. Our investigation was focused here on some properties useful when it comes to aggregation of vectors of different lengths. We have developed conditions required for the OM3 operators to be zero-insensitive, F-insensitive, and F+sensitive.

It is worth mentioning that in many applications it is more natural to consider OM3 operators that are continuous. A sufficient condition for that is the continuity of $\mathbf{w}$. In such case, the theorems presented in this paper have much simpler form. An OM3 operator is F-insensitive if and only if $\mathsf{w}(x) \leq x$ for all $x \in \mathbb{I}$. On the other hand, we get F+sensitivity together with zero-insensitivity if and only if $\mathsf{w}(x) \geq x$ for all $x \in \mathbb{I}$ and $c_1 = \mathsf{w}(b)$. From this we easily get, quite surprisingly, that the only continuous OM3 operator that fulfills all the properties discussed here is the Max operator.

Please note that the famous Hirsch index $\mathsf{H}(\mathbf{x}) = \max\{i = 1, \ldots, |\mathbf{x}| : x_{(n-i+1)} \geq i\} = \bigvee_{i=1}^{n} \lfloor x_{(n-i+1)} \rfloor \wedge i$, which is the most widely used tool in scientometrics, is an OM3 operator ($\mathsf{w}(x) = \lfloor x \rfloor$, $c_i = i$) fulfilling $\mathscr{P}_{(\mathrm{F0})}$ (and $\mathscr{P}_{(\mathrm{a0})}$).

Moreover, it is easily seen that an OM3 operator in $\mathscr{P}_{(\mathrm{a0})}$ is asymptotically idempotent [14], iff $\mathsf{w}(x) = x$ and $c_i \to b$ as $i \to \infty$. Additionally, each such operator is effort-dominating [7].

In the second part of our contribution we are going to perform a simulation study to asses behavior of OM3 operators for samples following different distributions. It will turn out that in order to study the effects of input vectors' ranking by means of OM3 operators, it suffices to consider a fixed triangle of coefficients. This makes the construction of OM3 operators quite easy in practical applications. However, what is still left for further research, is the method of automated construction of these operators.

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