Piecewise Linear Approximation of Fuzzy Numbers Preserving the Support and Core

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Abstract. A reasonable approximation of a fuzzy number should have a simple membership function, be close to the input fuzzy number, and should preserve some of its important characteristics. In this paper we suggest to approximate a fuzzy number by a piecewise linear 1-knot fuzzy number which is the closest one to the input fuzzy number among all piecewise linear 1-knot fuzzy numbers having the same core and the same support as the input. We discuss the existence of the approximation operator, show algorithms ready for the practical use and illustrate the considered concepts by examples. It turns out that such an approximation task may be problematic.

Keywords: Approximation of fuzzy numbers; core; fuzzy number; piecewise linear approximation; support.

1 Introduction

Complicated membership functions generate many problems in processing imprecise information modeled by fuzzy numbers including problems with calculations, computer implementation, etc. Moreover, handling complex membership functions entails difficulties in interpretation of the results too. This is the reason why a suitable approximation of fuzzy numbers is so important. So we usually try to substitute the original "input" membership functions by the "output" which is simpler or more regular and hence more convenient for further tasks. We expect that a desired approximation will reveal the following priorities:

(P.1) simplicity of a membership function,
(P.2) closeness to the input fuzzy number,
(P.3) preservation of some important characteristics.

The simplest possible shape of a membership function is acquired by linear sides. Hence, (P.1) is fulfilled by the trapezoidal approximation. One may do it in many ways but it seems that the desired approximation output should be as close as possible to the input. Thus, (P.1) and (P.2) may lead to the approximation of a fuzzy number by the closest trapezoidal one. However, such an approximation does not guarantee automatically any other interesting properties. Therefore, we often look for the approximation that has some additional properties like the invariance of the expected interval (see, e.g. [1, 6, 5, 8]). It seems that the core and the support belong to the most important characteristics of fuzzy numbers. It is quite obvious since these very sets are the only ones which are connected with our "sure" knowledge. Actually, the core contains all the points which surely belong to the fuzzy set under study. On the other hand, the complement of the support consists of the points that surely do not belong to given fuzzy set. The belongingness of all other points to the fuzzy set under discussion is just a matter of degree described quantitatively by the membership function. Hence, one may easily agree that both the support and core play a key role in fuzzy set analysis. However, if we try to approximate a fuzzy number by a trapezoidal one that preserves both the support and core of the input, the approximation problem simplifies too much since we obtain the unique solution just by joining the borders of the support and core by the straight lines. Unfortunately, the output of such approximation may be significantly distant from the input. The way out from this dilemma is to consider the approximation by a trapezoidal fuzzy number which is as close as possible to the input and preserves either the core or the support. This way we obtain a procedure which fulfills all the desired conditions (P.1)-(P.3). However, one may easily indicate examples where the output of the approximation with fixed core has the support significantly different than the support of the input. And conversely, the output of the approximation with fixed support may have the core significantly different than the core of the input.

This discussion shows that usually we cannot obtain a satisfying trapezoidal approximation of an arbitrary fuzzy number that fulfills the nearness criterion and preserves both the support and core. In this paper we propose to consider the 1-knot piecewise linear fuzzy numbers (see [2]) as a reasonable solution of the approximation problem satisfying requirements (P.1)-(P.3). More precisely, we suggest to approximate a fuzzy number by the closest piecewise linear 1-knot fuzzy number having the same core and the same support as the input.

2 Piecewise linear 1-knot fuzzy numbers

Fuzzy numbers are particular cases of fuzzy sets of the real line. The membership function of a fuzzy number A is given by:

$$A(x) = \begin{cases} 0 & \text{if} \quad x < a_1, \\ l_A(x) & \text{if} \quad a_1 \le x < a_2, \\ 1 & \text{if} \quad a_2 \le x \le a_3, \\ r_A(x) & \text{if} \quad a_3 < x \le a_4, \\ 0 & \text{if} \quad x > a_4, \end{cases} \tag{1}$$

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where $a_1, a_2, a_3, a_4 \in \mathbb{R}$, $l_A : [a_1, a_2] \longrightarrow [0, 1]$ is a nondecreasing upper semicontinuous function, $l_A(a_1) = 0$, $l_A(a_2) = 1$, called the *left side* of the fuzzy number, and $r_A : [a_3, a_4] \longrightarrow [0, 1]$ is a nonincreasing upper semicontinuous function, $r_A(a_3) = 1$, $r_A(a_4) = 0$, called the *right side* of the fuzzy number A. The α-*cut* of A, $\alpha \in (0, 1]$, is a crisp set defined as: $A_\alpha = \{x \in \mathbb{R} : A(x) \geq \alpha\}$. The *support* or 0-cut, A_0, of a fuzzy number is defined as

$$\mathrm{supp}(A) = A_0 = \overline{\{x \in \mathbb{R} : A(x) > 0\}}.$$

It is easily seen that for each $\alpha \in (0, 1]$ every α-cut of a fuzzy number is a closed interval $A_\alpha = [A_L(\alpha), A_U(\alpha)]$, where $A_L(\alpha) = \inf\{x \in \mathbb{R} : A(x) \geq \alpha\}$ and $A_U(\alpha) = \sup\{x \in \mathbb{R} : A(x) \geq \alpha\}$.

Moreover, if the sides of the fuzzy number A are strictly monotone, then A_L and A_U are inverse functions of l_A and r_A, respectively. The 1-cut of A will be called the core of A and we use the notation

$$A_1 = \mathrm{core}(A) = \{x \in \mathbb{R} : A(x) = 1\}.$$

¿From now on, we denote by $\mathbb{F}(\mathbb{R})$ the set of all fuzzy numbers. However, in practice, e.g. when calculations of arithmetic operations is performed, fuzzy numbers with simple membership functions are often preferred. The most commonly used subclass of $\mathbb{F}(\mathbb{R})$ is formed by so-called *trapezoidal fuzzy numbers*, i.e. fuzzy numbers with linear sides. The set of all trapezoidal fuzzy numbers is denoted by $\mathbb{F}^T(\mathbb{R})$. Trapezoidal fuzzy numbers are often used directly for modeling vague concepts or for approximating more complicated fuzzy numbers due to their simplicity. Unfortunately, in some situations such simple description may appear too limited. In some cases we are interested in specifying the membership function in one (or more) additional α-cuts other than 0 or 1. Thus in [2] a generalization of the trapezoidal fuzzy numbers was proposed by considering fuzzy numbers with piecewise linear side functions each consisting of two segments.

Definition 1. *For any fixed $\alpha_0 \in (0, 1)$ an α_0-piecewise linear 1-knot fuzzy number S is a fuzzy number with the following membership function*

$$S(x) = \begin{cases} 0 & \text{if} \quad x < s_1, \\ \alpha_0 \, \frac{x - s_1}{s_2 - s_1} & \text{if} \quad s_1 \leq x < s_2, \\ \alpha_0 + (1 - \alpha_0) \, \frac{x - s_2}{s_3 - s_2} & \text{if} \quad s_2 \leq x < s_3, \\ 1 & \text{if} \quad s_3 \leq x \leq s_4, \\ \alpha_0 + (1 - \alpha_0) \, \frac{s_5 - x}{s_5 - s_4} & \text{if} \quad s_4 < x \leq s_5, \\ \alpha_0 \, \frac{s_6 - x}{s_6 - s_5} & \text{if} \quad s_5 < x \leq s_6, \\ 0 & \text{if} \quad x > s_6, \end{cases}$$

where $\mathbf{s} = (s_1, \ldots, s_6)$ such that $s_1 \leq \cdots \leq s_6$.

Since any α_0-piecewise linear 1-knot fuzzy number is completely defined by its knot α_0 and six real numbers $s_1 \leq \cdots \leq s_6$, hence it will be denoted as $S = \mathrm{S}(\alpha_0, \mathbf{s})$. An example of an α_0-piecewise linear 1-knot fuzzy number is given in Fig. 1.

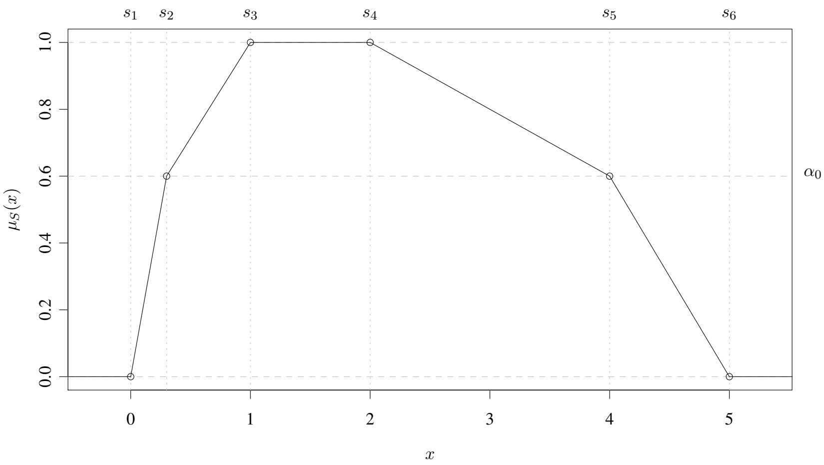

Fig. 1. The membership function of $S = \mathrm{S}(0.6, (0, 0.3, 1, 2, 4, 5))$.

Alternatively, an α_0-piecewise linear 1-knot fuzzy number may be defined using its α-cut representation, i.e.

$$S_L(\alpha) = \begin{cases} s_1 + (s_2 - s_1)\,\frac{\alpha}{\alpha_0} & \text{for} \quad \alpha \in [0, \alpha_0), \\ s_2 + (s_3 - s_2)\,\frac{\alpha - \alpha_0}{1 - \alpha_0} & \text{for} \quad \alpha \in [\alpha_0, 1] \end{cases} \tag{2}$$

and

$$S_U(\alpha) = \begin{cases} s_5 + (s_6 - s_5)\,\frac{\alpha_0 - \alpha}{\alpha_0} & \text{for} \quad \alpha \in [0, \alpha_0), \\ s_4 + (s_5 - s_4)\,\frac{1 - \alpha}{1 - \alpha_0} & \text{for} \quad \alpha \in [\alpha_0, 1]. \end{cases} \tag{3}$$

Let us denote the set of all such fuzzy numbers by $\mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})$. By setting $\mathbb{F}^{\pi(0)}(\mathbb{R}) = \mathbb{F}^{\pi(1)}(\mathbb{R}) := \mathbb{F}^T(\mathbb{R})$ we also include the cases $\alpha_0 \in \{0, 1\}$. Please note that the inclusion $\mathbb{F}^T(\mathbb{R}) \subseteq \mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})$ holds for any $\alpha_0 \in [0, 1]$.

Moreover, to simplify notation, let $\mathbb{F}^{\pi[a,b]}(\mathbb{R})$ denote the set of all α-piecewise linear 1-knot fuzzy numbers, where $\alpha \in [a, b]$ for some $0 \le a \le b \le 1$, i.e.

$$\mathbb{F}^{\pi[a,b]}(\mathbb{R}) := \bigcup_{\alpha \in [a,b]} \mathbb{F}^{\pi(\alpha)}(\mathbb{R}).$$

In many problems an adequate metric over the space of fuzzy numbers should be considered. The flexibility of the space of fuzzy numbers allows for the construction of many types of metric structures over this space. In the area of fuzzy number approximation the most suitable metric is an extension of the *Euclidean* (L^2) *distance* d defined by (see, e.g., [4])

$$d^2(A, B) = \int_0^1 (A_L(\alpha) - B_L(\alpha))^2 d\alpha + \int_0^1 (A_U(\alpha) - B_U(\alpha))^2 d\alpha. \tag{4}$$

3 Approximation problem

Let us consider any fuzzy number $A \in \mathbb{F}(\mathbb{R})$. Suppose we want to approximate A by an α_0-piecewise linear 1-knot fuzzy number S. In [2] the piecewise linear 1-knot fuzzy number approximation is broadly discussed. In this paper we suggest another type of approximation. Keeping in mind postulates (P.1)–(P.3), our goal now is to find the approximation which fulfills the following requirements:

1. Indicate the optimal knot α_0 for the piecewise linear 1-knot fuzzy number approximation of A, i.e. we are looking for the solution $S(A)$ in $\mathbb{F}^{\pi[0,1]}(\mathbb{R})$.
2. The solution should fulfill the so-called nearness criterion (see [6]), i.e. for any fuzzy number A the solution $S(A)$ should be the α_0-piecewise linear 1-knot fuzzy number nearest to A with respect to some predetermined metric. In our case we consider the distance d given by (4).
3. The solution should preserve the core and the support of A.

More formally, we are looking for such $S^* = S^*(A) \in \mathbb{F}^{\pi[0,1]}(\mathbb{R})$ that

$$d(A, S^*) = \min_{S \in \mathbb{F}^{\pi[0,1]}(\mathbb{R})} d(A, S), \tag{5}$$

which satisfies the following constraints:

$$\operatorname{core}(S^*) = \operatorname{core}(A), \tag{6}$$

$$\operatorname{supp}(S^*) = \operatorname{supp}(A). \tag{7}$$

At first, let us investigate whether the above problem has at least one solution for every $A \in \mathbb{F}(\mathbb{R})$. For that we will use the property that the space $(F(\mathbb{R}), d, +, \cdot)$ can be embedded in the Hilbert space $\left(L^2[0,1] \times L^2[0,1], \widetilde{d}, \oplus, \odot\right)$ (see e.g. [2]). Therefore, we have $d(A,B) = \widetilde{d}(A,B)$, $A + B = A \oplus B$ and $\lambda \cdot A = \lambda \odot A$, for all $A, B \in \mathbb{F}(\mathbb{R})$ and $\lambda \in [0, \infty)$. By Proposition 4 in [2] it is known that $\mathbb{F}^{\pi[0,1]}(\mathbb{R})$ is a closed subset of $L^2[0,1] \times L^2[0,1]$ in the topology generated by $\widetilde{d}$. Unfortunately, it may happen that the set

$$CS(A) = \left\{S \in \mathbb{F}^{\pi[0,1]}(\mathbb{R}) : \operatorname{core}(S) = \operatorname{core}(A), \operatorname{supp}(S) = \operatorname{supp}(A)\right\} \tag{8}$$

would not be closed in $\mathbb{F}^{\pi[0,1]}(\mathbb{R})$. Indeed, suppose that $A_\beta = [\beta^3, 1]$, $\beta \in [0,1]$. Then let us consider a sequence $(\mathrm{S}(\alpha_n, \mathbf{s}_n))_{n \geq 1}$, $\mathbf{s}_n = (s_{n,1}, ..., s_{n,6})$ in $CS(A)$, where for each $n \geq 1$ we have $\alpha_n = (n-1)/n$, $s_{n,1} = s_{n,2} = 0$ and $s_{n,3} = ... = s_{n,6} = 1$. It is immediate that $(\widetilde{d}) \lim_{n\to\infty} \mathrm{S}(\alpha_n, \mathbf{s}_n) = (d) \lim_{n\to\infty} \mathrm{S}(\alpha_n, \mathbf{s}_n) = [0,1]$ and since $\operatorname{core}([0,1]) \neq \operatorname{core}(A)$ it results that the set $CS(A)$ is not closed in $L^2[0,1] \times L^2[0,1]$, nor in $F(\mathbb{R})$. Therefore, it is an open question whether problem (5)-(7) has a solution for any $A \in \mathbb{F}(\mathbb{R})$.

Interestingly, the solution always exists if we consider a local approximation problem. Suppose that $0 < a < b < 1$ and let us consider the set $\mathbb{F}^{\pi[a,b]}(\mathbb{R}) = \{\mathrm{S}(\alpha, \mathbf{s}) \in \mathbb{F}^{\pi[0,1]}(\mathbb{R}) : a \leq \alpha \leq b\}$. Now let us consider the following set

$$CS_{a,b}(A) = \left\{S \in \mathbb{F}^{\pi[a,b]}(\mathbb{R}) : \operatorname{core}(S) = \operatorname{core}(A), \operatorname{supp}(S) = \operatorname{supp}(A)\right\}. \tag{9}$$

We are looking for such $S^* = S^*(A) \in CS_{a,b}(A)$ that

$$d(A, S^*) = \min_{S \in CS_{a,b}(A)} d(A, S). \tag{10}$$

Obviously, there is a sequence $(\mathrm{S}(\alpha_n, \mathbf{s}_n))_{n \geq 1}$, in $CS_{a,b}(A)$, such that

$$\lim_{n \to \infty} d(A, \mathrm{S}(\alpha_n, \mathbf{s}_n)) = \inf_{S \in CS_{a,b}(A)} d(A, S) := m. \tag{11}$$

Let $n_0 \in \mathbb{N}$ be such that $d(A, \mathrm{S}(\alpha_n, \mathbf{s}_n)) \leq m + 1$ for all $n \geq n_0$. This implies that $d(0, \mathrm{S}(\alpha_n, \mathbf{s}_n)) \leq d(0, A) + d(A, \mathrm{S}(\alpha_n, \mathbf{s}_n)) \leq d(0, A) + m + 1$ for all $n \geq n_0$. Therefore, the sequence $(\mathrm{S}(\alpha_n, \mathbf{s}_n))_{n \geq 1}$ is bounded with respect to metric d and hence with respect to $\widetilde{d}$. By Lemma 2 (iii) in [2] it results that each sequence $(c_{n,i})_{n \geq 1}$, $i = 1, \ldots, 8$, is bounded, where

$$\begin{aligned} s_{n,1} &= c_{n,1}, & s_{n,2} &= c_{n,2} \cdot \alpha_n + c_{n,1}, & s_{n,3} &= c_{n,3} + c_{n,4}, \\ s_{n,4} &= c_{n,7} + c_{n,8}, & s_{n,5} &= c_{n,5} + c_{n,6} \cdot \alpha_n, & s_{n,6} &= c_{n,5}. \end{aligned}$$

Without loss of generality let us suppose that $\lim_{n \to \infty} \alpha_n = \alpha_0$ (obviously we have $\alpha_0 \in [a, b]$) and $\lim_{n \to \infty} c_{n,i} = c_i$, $i = 1, \ldots, 8$. Letting $n \to \infty$ in the above equations and denoting $\mathbf{s} = (s_1, ..., s_6)$, where $s_i = \lim_{n \to \infty} s_{n,i}$, $i = 1, \ldots, 6$, it easily results that $\mathrm{S}(\alpha_0, \mathbf{s}) \in \mathbb{F}^{\pi[a,b]}(\mathbb{R})$. Then, since $\mathrm{S}(\alpha_n, \mathbf{s}_n) \in CS_{a,b}(A)$ for all $n \geq 1$, it follows that $s_{n,1} = A_L(0)$, $s_{n,3} = A_L(1)$, $s_{n,4} = A_U(1)$ and $s_{n,6} = A_U(0)$ and therefore we easily obtain that $\mathrm{S}(\alpha_0, \mathbf{s})$ preserves the core and support of A and hence $\mathrm{S}(\alpha_0, \mathbf{s}) \in CS_{a,b}(A)$. On the other hand, by Lemma 3 in [2] (making some suitable substitutions) we also obtain that $(\widetilde{d}) \lim_{n \to \infty} \mathrm{S}(\alpha_n, \mathbf{s}_n) = \mathrm{S}(\alpha_0, \mathbf{s})$. This property together with relation (11) and the continuity of d, implies that $d(A, \mathrm{S}(\alpha_0, \mathbf{s})) = m$. Hence we have just proved that problem (10) has at least one solution. Note that one can easily prove that $CS_{a,b}(A)$ is not convex in $L^2[0,1] \times L^2[0,1]$ which means that the solution of problem (10) may not be unique. All these results are summarized in the following theorem.

Theorem 1. *If $A \in \mathbb{F}(\mathbb{R})$ and $0 < a < b < 1$, then there exists at least one element $S^* = S^*(A) \in \mathbb{F}^{\pi[a,b]}(\mathbb{R})$ such that $d(A, S^*) = \min_{S \in CS_{a,b}(A)} d(A, S)$.*

4 Algorithm

Let us show how to find a solution to problem (10). We have to minimize the function

$$\begin{aligned} f(\alpha, x, y) = & \int_0^\alpha \left(A_L(\beta) - \left(A_L(0) + (x - A_L(0)) \cdot \frac{\beta}{\alpha} \right) \right)^2 d\beta \\ & + \int_\alpha^1 \left(A_L(\beta) - \left(x + (A_L(1) - x) \cdot \frac{\beta - \alpha}{1 - \alpha} \right) \right)^2 d\beta \\ & + \int_0^\alpha \left(A_U(\beta) - \left(y + (A_U(0) - y) \cdot \frac{\alpha - \beta}{\alpha} \right) \right)^2 d\beta \\ & + \int_\alpha^1 \left(A_U(\beta) - \left(A_U(1) + (y - A_U(1)) \cdot \frac{1 - \beta}{1 - \alpha} \right) \right)^2 d\beta \end{aligned}$$

subject to $A_L(0) \le x \le A_L(1)$ and $A_U(1) \le y \le A_U(0)$.

This problem may have more than one solution and, in addition, it seems to be difficult to be solved analytically in this form since the equation $f'_\alpha(\alpha, x, y) = 0$ cannot be solved in general as we are forced to work with functions where we cannot separate α from the integral. Therefore, we will start by considering the knot $\alpha = \alpha_0$ being fixed. For some $\alpha_0 \in (0,1)$ we want to minimize the function $g_{\alpha_0}(x, y) = f(\alpha_0, x, y)$ with the same restrictions as above. Obviously we can split this problem into two independent sub-problems. Firstly, we have to minimize the function

$$x \mapsto \int_0^{\alpha_0} \left(A_L(\beta) - \left(A_L(0) + (x - A_L(0)) \cdot \frac{\beta}{\alpha_0} \right) \right)^2 d\beta + \int_{\alpha_0}^1 \left(A_L(\beta) - \left(x + (A_L(1) - x) \cdot \frac{\beta - \alpha_0}{1 - \alpha_0} \right) \right)^2 d\beta$$

on the interval $[A_L(0), A_L(1)]$ and then we have to minimize the function

$$y \mapsto \int_0^{\alpha_0} \left(A_U(\beta) - \left(y + (A_U(0) - y) \cdot \frac{\alpha_0 - \beta}{\alpha_0} \right) \right)^2 d\beta + \int_{\alpha_0}^1 \left(A_U(\beta) - \left(A_U(1) + (y - A_U(1)) \cdot \frac{1 - \beta}{1 - \alpha_0} \right) \right)^2 d\beta$$

on the interval $[A_U(1), A_U(0)]$. Obviously, the above functions are quadratic functions of one variable and after some simple calculations we obtain their unique minimum points on $\mathbb{R}$ as

$$x_m = 3 \int_0^{\alpha_0} \left(A_L(\beta) - A_L(0) \cdot \frac{\alpha_0 - \beta}{\alpha_0} \right) \cdot \frac{\beta}{\alpha_0} d\beta + 3 \int_{\alpha_0}^1 \left(A_L(\beta) - A_L(1) \cdot \frac{\beta - \alpha_0}{1 - \alpha_0} \right) \cdot \frac{1 - \beta}{1 - \alpha_0} d\beta$$

and

$$y_m = 3 \int_0^{\alpha_0} \left(A_U(\beta) - A_U(0) \cdot \frac{\alpha_0 - \beta}{\alpha_0} \right) \cdot \frac{\beta}{\alpha_0} d\beta + 3 \int_{\alpha_0}^1 \left(A_U(\beta) - A_U(1) \cdot \frac{\beta - \alpha_0}{1 - \alpha_0} \right) \cdot \frac{1 - \beta}{1 - \alpha_0} d\beta.$$

¿From here we easily obtain the solutions of our two sub-problems as

$$x_0 = \begin{cases} A_L(0) & \text{if} \quad x_m < A_L(0), \\ A_L(1) & \text{if} \quad x_m > A_L(1), \\ x_m & \text{if} \quad A_L(0) \le x_m \le A_L(1) \end{cases} \tag{12}$$

and

$$y_0 = \begin{cases} A_U(1) & \text{if} \quad y_m < A_U(1), \\ A_U(0) & \text{if} \quad y_m > A_U(0), \\ y_m & \text{if} \quad A_U(1) \le y_m \le A_U(0). \end{cases} \tag{13}$$

When a computer implementation is needed, in most of the cases, x_m and y_m may be easily calculated via numeric integration, cf. [3].

In conclusion, we have just proved for fixed α the existence and uniqueness of the piecewise linear 1-knot approximation which preserves the core and the support. More exactly we have the following approximation result.

Theorem 2. *Suppose that* $\alpha_0 \in (0,1)$ *and for some fuzzy number* A *let us define the set*

$$CS_{\alpha_0}(A) = \{S \in \mathbb{F}^{\pi(\alpha_0)}(\mathbb{R}) : \mathrm{core}(S) = \mathrm{core}(A) \textit{ and } \mathrm{supp}(S) = \mathrm{supp}(A)\}.$$

Then there exists a unique best approximation (with respect to metric d*) of* A *relatively to the set* $CS_{\alpha_0}(A)$*. This approximation is* $S_{\alpha_0}(A) = \mathbf{S}(\alpha_0, \mathbf{s}(A))$*,* $\mathbf{s}(A) = (s_1(A), ..., s_6(A))$*, where*

$$\begin{array}{lll} s_1(A) = A_L(0), & s_2(A) = x_0, & s_3(A) = A_L(1), \\ s_4(A) = A_U(1), & s_5(A) = y_0, & s_6(A) = A_U(0), \end{array}$$

and x_0,y_0 *are given by (12) and (13) respectively.*

We will use the previous theorem to approach a solution $S^*(A) \in CS_{a,b}(A)$ of problem (10). We will construct a sequence $(S_{\alpha_n}(A))_{n\geq 1}$ in $CS_{a,b}(A)$ such that $(d)\lim_{n\to\infty} S_{\alpha_n}(A) = S^*(A)$. Here, $S_{\alpha_n}(A)$ is the unique best approximation of A relatively to the set $CS_{\alpha_n}(A)$.

5 Some numerical examples

Example 1. Consider a fuzzy number A with supp $= [0,5]$, core $= [3,4]$ and $A_L(\alpha) = 3\,\mathtt{qbeta}_A(\alpha; 2, 1)$, $A_U(\alpha) = 5 - \alpha^3$, where $\mathtt{qbeta}(x; a, b)$ denotes the quantile function of the Beta distribution $B(a, b)$. Let $\alpha_0 = 0.2$. The best piecewise linear approximation $A'_{0.2}$ of A preserving the support and core is defined by $\mathbf{s}' = (0, 1.53, 3, 4, 5, 5)$. We have $d(A, A'_{0.2}) \simeq 0.212$. On the other hand, for the best piecewise linear approximation $A''_{0.2}$ obtained using algorithm from [2] and given by $\mathbf{s}'' = (0.34, 1.46, 3.09, 4.26, 5.08, 5.08)$, we get $d(A, A''_{0.2}) \simeq 0.105$. The discussed fuzzy numbers are depicted in Fig. 2a.

Given a method to obtain $S^*_{\alpha_0}(A)$, i.e. best α_0-piecewise linear approximation of A preserving its support and core, we may find $S^*(A)$ using some general one-dimensional optimization technique, like the Brent algorithm implemented in R' `optimize()` function. This may be done for a fixed fuzzy number A by finding the argument minimizing the function $D_A(\alpha) = d(A, S^*_\alpha(A))$, $\alpha \in [0,1]$.

Example 2. Let us go back to the fuzzy number A from Example 1. Fig. 2b depicts the distance function D_A. By applying the Brent algorithm we find that for $\alpha^* \simeq 0.3$ we get the solution to our problem, with $d(A, S^*_{\alpha^*}) \simeq 1.999$.

Example 3. Consider a fuzzy number B with supp $= [-1,1]$ and core $= \{0\}$, $B_L(\alpha) = \mathtt{qbeta}(\alpha; 2, 2)$, $B_U(\alpha) = 1 - \mathtt{qbeta}(\alpha; 2, 2)$. Fig. 3a shows the distance function D_B having two minima near 0 and 1 that are hardly to find numerically.

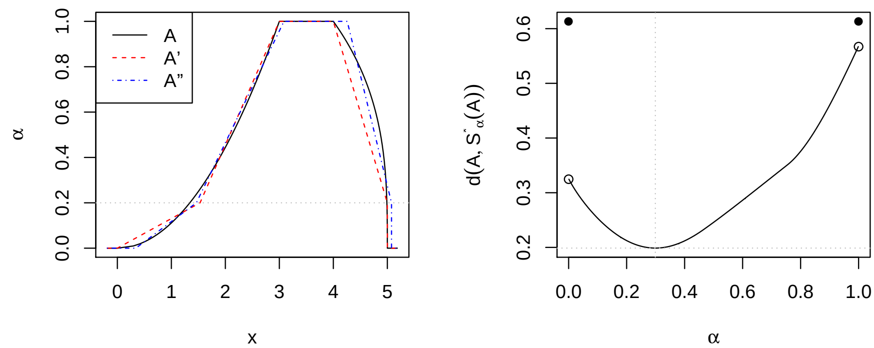

Fig. 2. (a) Fuzzy number A from Example 1 and its best approximation A' preserving its support and core as well as its best approximation $A''$with no such constraints; (b) distance function $D_A(\alpha)$

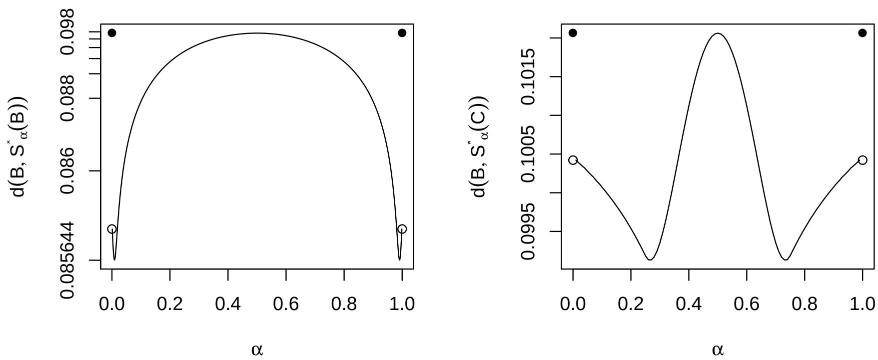

Fig. 3. (a) Distance function $D_B(\alpha)$ for a fuzzy number B from Example 3 (b) Distance function $D_C(\alpha)$ for a fuzzy number C from Example 4

Example 4. Consider a fuzzy number C with supp $= [0, 1]$ and core $= \{1\}$ with

$$C_L(\alpha) = \begin{cases} \alpha & \text{for} \quad \alpha \in [0, 0.25), \\ 0.5 & \text{for} \quad \alpha \in [0.25, 0.75], \\ \alpha & \text{for} \quad \alpha \in (0.75, 1]. \end{cases}$$

The distance function D_C is depicted in Fig. 3b. It has two minima at $\alpha = 0.25$ and $\alpha = 0.75$.

Moreover, an example may easily be constructed for which D_X has a local minimum that is not its global minimum. This information is important when using numerical optimization techniques, as an algorithm may fall into a suboptimal solution.

6 Conclusions

In this paper we have considered a fuzzy number approximation by a piecewise linear 1-knot fuzzy number which is the closest one to the input fuzzy number among all piecewise linear 1-knot fuzzy numbers having the same core and the same support as the input. We have indicated and discussed problems related to the existence and uniqueness of the solution.

One may easily notice that the approximation method suggested in this paper works nicely if the input fuzzy number is more or less symmetrical. However, if the left and the right side of a fuzzy number differ a lot, e.g. one is convex while the other is concave, our approximation method is not so convincing. It seems that in such a case a much more natural approach is to treat both sides of a fuzzy number separately.

Moreover, for particular fuzzy numbers (e.g. when the derivative in the neighborhood of the borders of the support or core is close to 0) it might be reasonable to consider weakened restrictions on the support and core than (6)–(7), just like suggested in [7], i.e. $\mathrm{supp}(S(A)) \subseteq \mathrm{supp}(A)$ and $\mathrm{core}(A) \subseteq \mathrm{core}(S(A))$.

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