Nearest piecewise linear approximation of fuzzy numbers

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Abstract

The problem of the nearest approximation of fuzzy numbers by piecewise linear 1-knot fuzzy numbers is discussed. By using 1-knot fuzzy numbers one may obtain approximations which are simple enough and flexible to reconstruct the input fuzzy concepts under study. They might be also perceived as a generalization of the trapezoidal approximations. Moreover, these approximations possess some desirable properties. Apart from theoretical considerations approximation algorithms that can be applied in practice are also given.

Keywords: Approximation of fuzzy numbers; fuzzy number; piecewise linear approximation.

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1. Introduction

Fuzzy numbers appear as the most popular family of fuzzy sets useful both for theoretical considerations as well as diverse practical applications.

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However, complicated membership functions have many drawbacks in processing imprecise information modeled by fuzzy numbers including problems with calculations, computer implementation, etc. Moreover, handling too complex membership functions entails difficulties in interpretation of the results too. This is the reason that a suitable approximation of fuzzy numbers is so important. One just tries to substitute the original "input" membership functions by another "outputs" which are simpler or more regular and hence more convenient for further tasks.

The most restrictive way of simplifying fuzzy numbers is their defuzzification, substituting a fuzzy set by a single real number. Unfortunately, this results in too much loss of information. Hence most researchers are nowadays interested in the interval (see e.g. [13, 18, 21]) or trapezoidal approximation which enables simple calculations, data processing and management of uncertainty. In particular, the trapezoidal approximation preserving the expected interval was suggested in [22] and then improved and further developed in [4, 5, 9, 14, 19, 20, 23, 34, 35]. Weighted trapezoidal approximation was considered in [3, 24, 25, 36, 37, 38]. In [12] some connections between trapezoidal approximation and aggregation were discussed. Other aspects of the trapezoidal approximation and its generalizations could be found in [1, 2, 6, 7, 10, 11, 26, 27].

It seems that the trapezoidal approximation may be satisfactory for many real-life cases. However, in some situations this kind of approximation may also be too restrictive. Indeed, when approximating arbitrary fuzzy numbers by trapezoidal ones we generally take care for the core and support of a fuzzy number (i.e. for values that surely belong or do not belong at all to the set under study), while the sides of a fuzzy number corresponding to all intermediate degrees of membership are linearized. This approach may not be suitable if we are also interested in focusing on some specified degree of uncertainty except for 0 or 1. Therefore, sometimes it would be desirable to approximate a fuzzy number by one that puts more attention to these intermediate uncertainty levels between 0 and 1. Hence, keeping in mind our general tendency to move towards simplicity we propose a generalization of the trapezoidal fuzzy numbers by considering fuzzy numbers with piecewise linear sides each consisting of two segments. Such a fuzzy number, called a piecewise linear 1-knot fuzzy number, is completely characterized by six points on the real line. Using 1-knot fuzzy numbers we may obtain approximations which are still simple but more flexible for reconstructing the input fuzzy concepts.

As a matter of fact, piecewise linear fuzzy quantities were studied much earlier by a few researchers. For example, Baekeland and Kerre [8] investigated the mathematical properties of piecewise linear fuzzy quantities, later implemented in expert systems and in fuzzy database systems [31]. Piecewise linear membership functions were successfully applied to fuzzy mathematical programming problems (see, e.g. [29, 32]).

The paper is organized as follows. In Section 2 we recall basic information on fuzzy numbers and some tools convenient for dealing with fuzzy numbers. Moreover, we define so-called piecewise linear 1-knot fuzzy numbers which are of central interest in this paper. In Sec. 3 we present some theoretical results on convergence in the Hilbert space, in which the space of fuzzy numbers may be embedded, and on the family of piecewise linear 1-knot fuzzy numbers. Since these results are rather auxiliary the proofs of the theorems are placed in the Appendix. The main goal of the paper, i.e. the piecewise linear 1-knot fuzzy number approximation with a fixed knot, is specified and broadly discussed in Sec. 4. There we show not only the existence of the solution of the nearest L^2-approximation problem but we also deliver two exact algorithms producing the desired approximations. However, we consider the properties of the approximation operator. Finally, in Sec. 5 we discuss a more general problem of finding the optimal knot for the piecewise linear 1-knot fuzzy number approximation of a fuzzy number.

The proposed approximation algorithms were implemented in the `FuzzyNumbers` [16] package for the `R` environment [28].

2. Preliminaries

Fuzzy numbers are the most commonly used fuzzy subsets of the real line. The membership function of a fuzzy number A is given by:

$$\mu_A(x) = \begin{cases} 0 & \text{if} \quad x < a_1, \\ l_A(x) & \text{if} \quad a_1 \le x < a_2, \\ 1 & \text{if} \quad a_2 \le x \le a_3, \\ r_A(x) & \text{if} \quad a_3 < x \le a_4, \\ 0 & \text{if} \quad x > a_4, \end{cases} \tag{1}$$

where $a_1, a_2, a_3, a_4 \in \mathbb{R}$, $l_A : [a_1, a_2] \longrightarrow [0, 1]$ is a nondecreasing upper semicontinuous function such that $l_A(a_1) = 0$, $l_A(a_2) = 1$, and $r_A : [a_3, a_4] \longrightarrow [0, 1]$ is a nonincreasing upper semicontinuous function fulfilling $r_A(a_3) = 1$,

$r_A(a_4) = 0$. l_A and r_A are called the *left* and *right sides* of A, respectively. For any $\alpha \in (0,1]$, the *α-cut* of a fuzzy number A is a crisp set defined as

$$A_\alpha = \{x \in \mathbb{R} : A(x) \geq \alpha\}.$$

The *support* or 0-cut, A_0, of a fuzzy number is defined as

$$A_0 = \overline{\{x \in \mathbb{R} : A(x) > 0\}}.$$

It is easily seen that for each $\alpha \in [0,1]$ every α-cut of a fuzzy number is a closed interval

$$A_\alpha = [A_L(\alpha), A_U(\alpha)],$$

where $A_L(\alpha) = \inf\{x \in \mathbb{R} : A(x) \geq \alpha\}$ and $A_U(\alpha) = \sup\{x \in \mathbb{R} : A(x) \geq \alpha\}$. If the sides of the fuzzy number A are strictly monotone then A_L and A_U are inverse functions of l_A and r_A, respectively. Two fuzzy numbers A and B are equal if $A_L(\alpha) = B_L(\alpha)$ and $A_U(\alpha) = B_U(\alpha)$ almost everywhere, $\alpha \in [0,1]$. The set of all fuzzy numbers will be denoted by $\mathbb{F}(\mathbb{R})$.

In a family of fuzzy numbers we may define addition and scalar multiplication (see e.g. [15]). Let $A, B \in \mathbb{F}(\mathbb{R})$, $\alpha \in [0,1]$ and $\lambda \in \mathbb{R}$. Then the sum of two fuzzy numbers A and B is a fuzzy number $A + B$ with the α-cuts

$$(A+B)_\alpha = A_\alpha + B_\alpha = [A_L(\alpha) + B_L(\alpha), A_U(\alpha) + B_U(\alpha)],$$

while the scalar multiplication $\lambda \cdot A$ is defined by

$$(\lambda \cdot A)_\alpha = \lambda A_\alpha = \begin{cases} [\lambda A_L(\alpha), \lambda A_U(\alpha)], & \text{if } \lambda \geq 0, \\ [\lambda A_U(\alpha), \lambda A_L(\alpha)], & \text{if } \lambda < 0. \end{cases}$$

In practical problems like solving fuzzy equations, data analysis or ranking fuzzy numbers, an adequate metric over the space of fuzzy numbers should be considered. The flexibility of the space of fuzzy numbers allows for the construction of many types of metric structures over this space. In the area of fuzzy number approximation the most suitable metric is an extension of the *Euclidean* (L^2) *distance* d defined by (see, e.g. [17])

$$d^2(A,B) = \int_0^1 (A_L(\alpha) - B_L(\alpha))^2 d\alpha + \int_0^1 (A_U(\alpha) - B_U(\alpha))^2 d\alpha. \tag{2}$$

Let us also recall that if (X, d_1) and (Y, d_2) are metric spaces then a function $f : X \to Y$ is called a *Lipschitz function* if there exists a real positive constant C such that

$$d_2(f(x), f(x')) \leq C\, d_1(x, x')$$

for all $x, x' \in X$. It is well-known that Lipschitz functions are continuous. If we have $C \leq 1$ in the above inequality then f is called a *nonexpansive function*.

Although family $\mathbb{F}(\mathbb{R})$ is quite rich and consists of fuzzy numbers with diverse membership functions, fuzzy numbers with simpler membership functions are often preferred in practice. The most commonly used subclass of $\mathbb{F}(\mathbb{R})$ is formed by so-called *trapezoidal fuzzy numbers*, i.e. fuzzy numbers with linear sides. Thus a membership function of a trapezoidal fuzzy number is given by

$$\mu_T(x) = \begin{cases} 0 & \text{if} \quad x < t_1, \\ \frac{x-t_1}{t_2-t_1} & \text{if} \quad t_1 \leq x < t_2, \\ 1 & \text{if} \quad t_2 \leq x \leq t_3, \\ \frac{t_4-x}{t_4-t_3} & \text{if} \quad t_3 < x \leq t_4, \\ 0 & \text{if} \quad x > t_4, \end{cases}$$

where $t_1 \leq t_2 \leq t_3 \leq t_4$. Since the membership function of a trapezoidal fuzzy number T is completely defined by these four real numbers we denote it usually as $T = \mathrm{T}(t_1, t_2, t_3, t_4)$. It is easy to prove that

$$\begin{aligned} T_L(\alpha) &= t_1 + (t_2 - t_1)\alpha, \\ T_U(\alpha) &= t_4 - (t_4 - t_3)\alpha. \end{aligned}$$

The set of all trapezoidal fuzzy numbers is denoted by $\mathbb{F}^T(\mathbb{R})$.

Trapezoidal fuzzy numbers are often used directly for modeling vague concepts or for approximating more complicated fuzzy numbers due to their simplicity. Unfortunately, in some situations such simple description may appear too limited. In some cases we are interested in specifying the membership function in one (or more) additional α-cuts other than 0 or 1. Thus in this paper we propose a generalization of the trapezoidal fuzzy numbers by considering fuzzy numbers with piecewise linear side functions each consisting of two segments.

Definition 1. *For any fixed $\alpha_0 \in (0, 1)$ an α_0-piecewise linear 1-knot fuzzy* *number S is a fuzzy number with the following membership function*

$$\mu_S(x) = \begin{cases} 0 & \text{if } x < s_1, \\ \alpha_0 \frac{x-s_1}{s_2-s_1} & \text{if } s_1 \le x < s_2, \\ \alpha_0 + (1-\alpha_0)\frac{x-s_2}{s_3-s_2} & \text{if } s_2 \le x < s_3, \\ 1 & \text{if } s_3 \le x \le s_4, \\ \alpha_0 + (1-\alpha_0)\frac{s_5-x}{s_5-s_4} & \text{if } s_4 < x \le s_5, \\ \alpha_0 \frac{s_6-x}{s_6-s_5} & \text{if } s_5 < x \le s_6, \\ 0 & \text{if } x > s_6, \end{cases}$$

where $\mathbf{s} = (s_1, \ldots, s_6)$ *such that* $s_1 \le \cdots \le s_6$.

Since any α_0-piecewise linear 1-knot fuzzy number is completely defined by its knot α_0 and six real numbers $s_1 \le \cdots \le s_6$ hence it will be denoted as $S = \mathrm{S}(\alpha_0, \mathbf{s})$. An example of the α_0-piecewise linear 1-knot fuzzy number is given in Fig. 1.

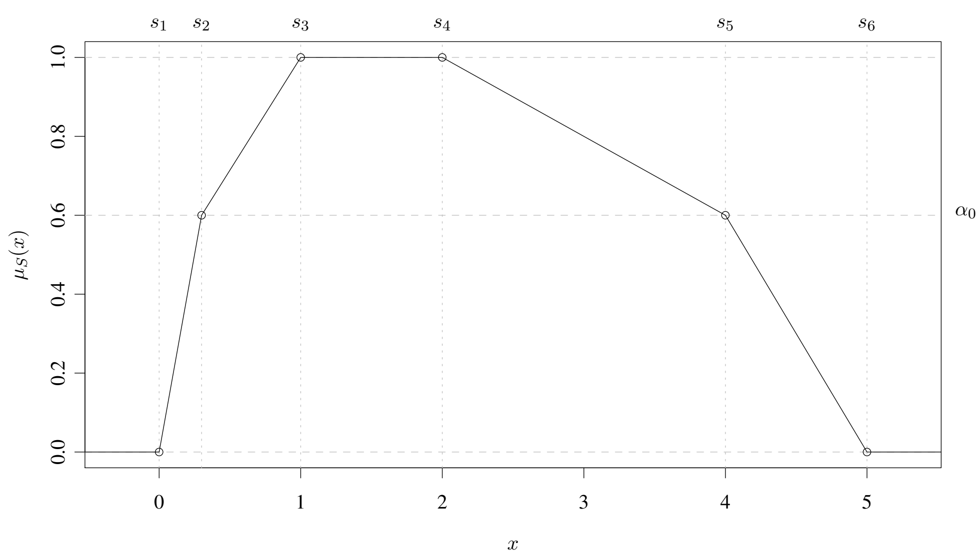

Figure 1: The membership function of $S = \mathrm{S}(0.6, (0, 0.3, 1, 2, 4, 5)))$.

Alternatively, α_0-piecewise linear 1-knot fuzzy number may be defined using its α-cut representation, i.e.

$$S_L(\alpha) = \begin{cases} s_1 + (s_2 - s_1)\frac{\alpha}{\alpha_0} & \text{for } \alpha \in [0, \alpha_0), \\ s_2 + (s_3 - s_2)\frac{\alpha-\alpha_0}{1-\alpha_0} & \text{for } \alpha \in [\alpha_0, 1] \end{cases} \tag{3}$$

and

$$S_U(\alpha) = \begin{cases} s_5 + (s_6 - s_5)\,\frac{\alpha_0-\alpha}{\alpha_0} & \text{for} \quad \alpha \in [0, \alpha_0), \\ s_4 + (s_5 - s_4)\,\frac{1-\alpha}{1-\alpha_0} & \text{for} \quad \alpha \in [\alpha_0, 1]. \end{cases} \tag{4}$$

Let us denote the set of all such fuzzy numbers by $\mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})$. By setting $\mathbb{F}^{\pi(0)}(\mathbb{R}) = \mathbb{F}^{\pi(1)}(\mathbb{R}) := \mathbb{F}^{T}(\mathbb{R})$ we also include the cases $\alpha_0 \in \{0, 1\}$. Please note that the inclusion $\mathbb{F}^{T}(\mathbb{R}) \subseteq \mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})$ holds for any $\alpha_0 \in [0, 1]$. Indeed, if $T = \mathbf{T}(t_1, t_2, t_3, t_4)$ is a trapezoidal fuzzy number and $\alpha_0 \in (0, 1)$ then we have $T = \mathbf{S}(\alpha_0, \mathbf{s})$ where $\mathbf{s} = (s_1, ..., s_6)$ and $s_1 = t_1, s_2 = t_1 + (t_2 - t_1)\alpha_0, s_3 = t_2, s_4 = t_3, s_5 = t_4 - (t_4 - t_3)\alpha_0, s_6 = t_4$.

Moreover, to simplify notation, let $\mathbb{F}^{\pi[a,b]}(\mathbb{R})$ denote the set of all α-piecewise linear 1-knot fuzzy numbers, where $\alpha \in [a, b]$ for some $0 \le a \le b \le 1$, i.e.

$$\mathbb{F}^{\pi[a,b]}(\mathbb{R}) := \bigcup_{\alpha \in [a,b]} \mathbb{F}^{\pi(\alpha)}(\mathbb{R}).$$

It is worth noting that in the case of α_0-piecewise linear 1-knot fuzzy numbers some calculations simplify, e.g. for $S' = \mathrm{S}(\alpha_0, \mathbf{s}')$ and $S'' = \mathrm{S}(\alpha_0, \mathbf{s}'')$ we obtain $S' + S'' = \mathrm{S}(\alpha_0, (s'_1 + s''_1, \ldots, s'_6 + s''_6))$, while $\lambda S = \mathrm{S}(\alpha_0, (\lambda s_1, \ldots, \lambda s_6))$ for $\lambda \ge 0$, and $\lambda S = \mathrm{S}(\alpha_0, (\lambda s_6, \ldots, \lambda s_1))$ otherwise.

3. Auxiliary results

Before we present the main results of this contribution let us briefly discuss some auxiliary topics that are necessary for our further considerations. In particular, we have to provide some basics from the theory of L^2 integrable functions. It is so because, as it is known from Yeh's papers [33, 36], we can embed $(\mathbb{F}(\mathbb{R}), d, +, \cdot)$ into the Hilbert space $(L^2[0,1] \times L^2[0,1], \widetilde{d}, \oplus, \odot)$, where $L^2[0,1]$ denotes the space of L^2 integrable functions on $[0,1]$ and for $\mathbf{f} = (f_1, f_2), \mathbf{g} = (g_1, g_2) \in L^2[0,1] \times L^2[0,1]$ we have

$$\widetilde{d}^2(\mathbf{f}, \mathbf{g}) = \int_0^1 (f_1(\alpha) - g_1(\alpha))^2\, d\alpha + \int_0^1 (f_2(\alpha) - g_2(\alpha))^2\, d\alpha. \tag{5}$$

while $A \oplus B = A + B$ and $\lambda \odot A = \lambda \cdot A$ for any $A, B \in \mathbb{F}(\mathbb{R})$ and $\lambda \in [0, \infty)$, as $(A_L, A_U), (B_L, B_U) \in L^2[0,1] \times L^2[0,1]$. Note that the inner product which generates $\widetilde{d}$ is given by

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_0^1 f_1(\alpha)\, g_1(\alpha)\, d\alpha + \int_0^1 f_2(\alpha)\, g_2(\alpha)\, d\alpha. \tag{6}$$

Further on if a sequence $(f_n)_{n\geq 1}$ of functions from $L^2[a,b]$ converges to $f \in L^2[a,b]$, we will use the notation $\|f_n - f\|_{[a,b]} \to 0$, where $\|\|$ denotes a usual L^2 norm. An immediate consequence is that we also have $\|f_n - f\|_{[c,d]} \to 0$ for any interval $[c,d] \subseteq [a,b]$. Note that we consider the restriction of the functions to the subinterval $[c,d]$ here but there is no need at all to change the notations.

If a sequence $((f_{n,1}, f_{n,2}))_{n\geq 1}$ with elements from the space $L^2[0,1] \times L^2[0,1]$ converges to $(f_1, f_2) \in L^2[0,1] \times L^2[0,1]$ with respect to the metric $\widetilde{d}$ then we will denote it by $\lim^{(\widetilde{d})}_{n\to\infty}(f_{n,1}, f_{n,2}) = (f_1, f_2)$ or alternatively $\|(f_{n,1}, f_{n,2}) \ominus (f_1, f_2)\|_{\widetilde{d}} \to 0$, where $(f_{n,1}, f_{n,2}) \ominus (f_1, f_2) = (f_{n,1} - f_1, f_{n,2} - f_2)$ for any n. It is immediate that this convergence property results in $\|f_{n,1} - f_1\|_{[0,1]} \to 0$ and $\|f_{n,2} - f_2\|_{[0,1]} \to 0$. In particular, if $(A_n)_{n\geq 1}$ is a sequence of fuzzy numbers such that $\lim_{n\to\infty} d(A_n, A) = 0$ then by the properties of the extended distance $\widetilde{d}$ we have $\lim_{n\to\infty} \widetilde{d}(A_n, A) = 0$, $\|A_{n,L} - A_L\|_{[0,1]} \to 0$ and $\|A_{n,U} - A_U\|_{[0,1]} \to 0$, where $A_{n,L}$ and $A_{n,U}$ denote functions which are the left and right arm of a fuzzy number A_n, respectively. Additionally, let us note that if $(\alpha_n)_{n\geq 1}$ and $(\beta_n)_{n\geq 1}$ are sequences of real numbers convergent to $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$, respectively, then it holds $\|(\alpha_n x + \beta_n) - (\alpha x + \beta)\|_{[a,b]} \to 0$ (as functions in argument x) for any compact interval $[a,b]$.

Now let us concentrate on the particular elements $\mathbf{f} = (f_1, f_2)$ of the space $L^2[0,1] \times L^2[0,1]$ such that

$$f_1(\alpha) = \begin{cases} c_1 + c_2\,\alpha & \text{if} \quad \alpha \in [0, \alpha_0], \\ c_3 + c_4\,\alpha & \text{if} \quad \alpha \in (\alpha_0, 1] \end{cases} \tag{7}$$

$$f_2(\alpha) = \begin{cases} c_5 + c_6\,\alpha & \text{if} \quad \alpha \in [0, \alpha_0], \\ c_7 + c_8\,\alpha & \text{if} \quad \alpha \in (\alpha_0, 1] \end{cases} \tag{8}$$

for some fixed $\alpha_0 \in [0,1]$ and $\mathbf{c} = (c_1, ..., c_8) \in \mathbb{R}^8$. For such objects we will use the notation $\mathbf{f} = (f_1, f_2) =: \mathrm{S}_e(\alpha_0, \mathbf{c})$. Moreover, for any fixed $0 \leq a \leq b \leq 1$ we will consider the spaces

$$\mathbb{S}_e^{\alpha_0}(\mathbb{R}) = \{\mathrm{S}_e(\alpha_0, \mathbf{c}) : \mathbf{c} \in \mathbb{R}^8\},$$

where $\alpha_0 \in [a,b]$, and

$$\mathbb{S}_e^{[a,b]}(\mathbb{R}) = \{\mathrm{S}_e(\alpha_0, \mathbf{c}) : \alpha_0 \in [a,b], \mathbf{c} \in \mathbb{R}^8\}.$$

Please note that $\mathbb{S}_e^{\alpha_0}(\mathbb{R}) \not\subset \mathbb{F}(\mathbb{R})$. However, for an α_0-piecewise linear 1-knot fuzzy number $S \in \mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})$ we easily obtain the representation given by (7)–(8) by solving the linear equations

$$\begin{aligned} s_1 &= c_1, &(9)\\ s_2 &= c_2\alpha_0 + c_1, &(10)\\ s_3 &= c_3 + c_4, &(11)\\ s_4 &= c_7 + c_8, &(12)\\ s_5 &= c_5 + c_6\alpha_0, &(13)\\ s_6 &= c_6. &(14) \end{aligned}$$

Thus we have $\mathbb{F}^{\pi(\alpha_0)} \subset \mathbb{S}_e^{\alpha_0}(\mathbb{R})$. Similarly, $\mathbb{F}^{\pi[a,b]} \subset \mathbb{S}_e^{[a,b]}(\mathbb{R})$ for any $0 \leq a \leq b \leq 1$. Therefore, many convergence properties in the spaces $\mathbb{S}_e^{\alpha_0}(\mathbb{R})$ and $\mathbb{S}_e^{[a,b]}(\mathbb{R})$ with respect to the metric $\widetilde{d}$ have corresponding results in the spaces $\mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})$ and $\mathbb{F}^{\pi[a,b]}$ with respect to the metric d.

Lemma 2. *Let $(\mathrm{S}_e(\alpha_n, \mathbf{c}_n))_{n\geq 1}$, $\mathbf{c}_n = (c_{n,1}, \ldots, c_{n,8})$, be a bounded sequence in the space $L^2[0,1] \times L^2[0,1]$ with respect to the metric $\widetilde{d}$. Then:*

- *(i) if there exists $a \in (0,1]$ such that $a \leq \alpha_n$ for any $n \geq 1$ then the sequences $(c_{n,i})_{n\geq 1}$ for $i \in \{1,2,5,6\}$ are all bounded;*
- *(ii) if there exists $b \in [0,1)$ such that $\alpha_n \leq b$ for any $n \geq 1$ then the sequences $(c_{n,i})_{n\geq 1}$ for $i \in \{3,4,7,8\}$ are all bounded;*
- *(iii) if there exist $a, b \in (0,1)$ such that $a \leq \alpha_n \leq b$ for any $n \geq 1$ then the sequences $(c_{n,i})_{n\geq 1}$ are bounded for any $i \in \{1,\ldots,8\}$.*

For the proof we refer the reader to Appendix A.

Please note that we cannot relax the assumptions $a, b \in (0,1)$ in the assertion (iii) of the previous lemma. The following illustration stands for a counterexample. Let us consider the sequence $(\mathbf{f}_n) = ((f_{n,1}, f_{n,2}))_{n\geq 1}$ such that:

$$\begin{aligned} f_{n,1}(\alpha) &= \begin{cases} -\sqrt{n} + \alpha\, n\sqrt{n} & \text{if} \quad \alpha \in [0, 1/n], \\ 0 & \text{if} \quad \alpha \in (1/n, 1], \end{cases} \\ f_{n,2}(\alpha) &= 0. \end{aligned}$$

We have (cf. Eqs. (7) and (8)) $c_{n,1} = -\sqrt{n}$, $c_{n,2} = n\sqrt{n}$, $c_{n,3} = \cdots = c_{n,8} = 0$, and $\alpha_n = 1/n$. We get $\|(\mathbf{f}_n)\|_{\widetilde{d}} = \sqrt{3}/3$, which implies that $(\mathbf{f}_n)_{n\geq 1}$ is bounded. However, the sequences $(c_{n,1})_{n\geq 1}$ and $(c_{n,2})_{n\geq 1}$ are unbounded. It means that if $\alpha_n \searrow 0$ (or there exists a subsequence $\alpha_{k_n} \searrow 0$) then assertion (iii) (or assertion (i)) from the above lemma does not hold in general for $(c_{n,i})_{n\geq 1}$, $i \in \{1, 2, 5, 6\}$. Similarly, if $\alpha_n \nearrow 1$ then assertion (iii) (or assertion (ii)) from the above lemma does not hold in general for $(c_{n,i})_{n\geq 1}$, $i \in \{3, 4, 7, 8\}$.

Lemma 3. *Let* $(\mathrm{S}_e(\alpha_n, \mathbf{c}_n))_{n\geq 1}$, $\mathbf{c}_n = (c_{n,1}, \ldots, c_{n,8})$, *be a sequence in the space* $L^2[0,1] \times L^2[0,1]$ *satisfying* $\alpha_n \to \alpha$ *and* $c_{n,i} \to c_i$ *for each* $i \in \{1, \ldots, 8\}$. *Then* $\|\mathrm{S}_e(\alpha_n, \mathbf{c}_n) - \mathrm{S}_e(\alpha, \mathbf{c})\|_{\widetilde{d}} \to 0$ *where* $\mathbf{c} = (c_1, \ldots, c_8)$.

For the proof we refer the reader to Appendix B.

Now we are ready to prove the most important results of this section.

Proposition 4. *The set* $\mathbb{F}^{\pi[0,1]}(\mathbb{R})$ *is a closed subset of the space* $L^2[0,1] \times L^2[0,1]$ *endowed with the topology generated by the metric* $\widetilde{d}$.

For the proof we refer the reader to Appendix C.

A similar proposition holds also for $\mathbb{F}^{\pi[a,b]}(\mathbb{R})$, $[a,b] \subseteq [0,1]$.

Proposition 5. *If* $0 \leq a \leq b \leq 1$ *are arbitrarily chosen then the set* $\mathbb{F}^{\pi[a,b]}(\mathbb{R})$ *is a closed subset of the space* $L^2[0,1] \times L^2[0,1]$ *endowed with the topology generated by the metric* $\widetilde{d}$.

For the proof we refer the reader to Appendix D.

Remark 6. If $0 \leq a < b \leq 1$ then $\mathbb{F}^{\pi[a,b]}(\mathbb{R})$ is not convex.

Proof. Consider $S_1 \in \mathbb{F}^{\pi(\alpha_1)}(\mathbb{R})$, and $S_2 \in \mathbb{F}^{\pi(\alpha_2)}(\mathbb{R})$ for $a < \alpha_1 < \alpha_2 < b$ such that $S_1, S_2 \notin \mathbb{F}^{T}(\mathbb{R})$. It is easily seen that $S_1 + S_2 \notin \mathbb{F}^{\pi[a,b]}(\mathbb{R})$, because the operation gives as a result a piecewise linear fuzzy number with 2 knots (instead of 1). Since the space $\mathbb{F}^{\pi[a,b]}(\mathbb{R})$ is nonadditive (strictly, not closed under addition) it is also not convex. $\square$

Finally, we can show that closedness holds also for $\mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})$, where α_0 is chosen arbitrarily. It should be stressed that the subset of interest is convex.

Proposition 7. *If* $\alpha_0 \in [0,1]$ *is arbitrarily chosen then the set* $\mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})$ *is a closed convex subset of the space* $L^2[0,1] \times L^2[0,1]$ *endowed with the topology generated by the metric* $\widetilde{d}$.

We omit the simple proof of the convexity property (one may easily obtain that $\mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})$ is a closed set by taking $a = b = \alpha_0$ and applying the previous proposition).

4. The best approximation for a fixed knot α_0

4.1. Existence and uniqueness

Let us consider any fuzzy number $A \in \mathbb{F}(\mathbb{R})$. Suppose we want to approximate A by an α_0-piecewise linear 1-knot fuzzy number S. Of course, one can do it in many ways thus some additional criteria should be established. In this section we assume the following requirements:

1. The α_0-cut representing the knot of the piecewise linear 1-knot fuzzy number approximating A is fixed, i.e. we are looking for the solution $S(A)$ in $\mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})$.
2. The solution should fulfill the so-called nearness criterion (see [22]), i.e. for any fuzzy number A the solution $S(A)$ should be the α_0-piecewise linear 1-knot fuzzy number nearest to A with respect to some predetermined metric. In our case we consider the distance d given by (2).

To sum up, our goal is as follows: For a fixed $\alpha_0 \in [0,1]$ and for any $A \in \mathbb{F}(\mathbb{R})$ we are looking for $S^*_{\alpha_0}(A) \in \mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})$ such that

$$d(A, S^*_{\alpha_0}(A)) = \min_{S \in \mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})} d(A, S). \tag{15}$$

The following theorem shows the existence of the result and its uniqueness in the problem of best approximation with respect to $\mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})$.

Theorem 8. *If* A *is an arbitrary fuzzy number and* $\alpha_0 \in [0,1]$ *then there exists a unique fuzzy number* $S^*_{\alpha_0}(A) \in \mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})$ *satisfying (15).*

Proof. Recall that $d(A,B) = \widetilde{d}(A,B)$ for all $B \in \mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})$. By Proposition 7, $\mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})$ is a closed convex subset of the Hilbert space $(L^2[0,1] \times L^2[0,1], \widetilde{d}, \oplus, \odot)$. Therefore, for any element from $L^2[0,1] \times L^2[0,1]$ there exists a unique best approximation relatively to the set $\mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})$ with respect to the metric $\widetilde{d}$, cf. [30, Theorem 4.10], which proves the theorem. $\square$

4.2. Derivation of the solution

Basing on the result presented above we can define an approximation operator $\Pi_{\alpha_0} : \mathbb{F}(\mathbb{R}) \to \mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})$ which assigns to a fuzzy number A its unique best approximation relatively to the space $\mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})$ with respect to the metric d, that is $\Pi_{\alpha_0}(A) = S^*_{\alpha_0}(A)$. In this section we provide algorithms to compute $\Pi_{\alpha_0}(A)$ for any fuzzy number A and then we investigate the properties of the Π_{α_0} operator. We restrict our considerations to $\alpha_0 \in (0,1)$ because for $\alpha_0 \in \{0,1\}$ the approximation operator Π_{α_0} becomes the trapezoidal approximation operator which was already studied in the recent literature (see e.g. [2, 5, 34]).

Let $\varkappa_1, \varkappa_2 \in L^2[0,1]$ denote the characteristic functions of the intervals $[0,\alpha_0]$ and $(\alpha_0,1]$, respectively. Additionally, let us consider the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_8$ in the Hilbert space $\left(L^2[0,1] \times L^2[0,1], \widetilde{d}\right)$, $\mathbf{v}_i = (v_{i,1}, v_{i,2})$ for $i \in \{1, \ldots, 8\}$, such that

$$\begin{array}{rclrcl}
v_{1,1}(\alpha) & = & \varkappa_1(\alpha), & v_{1,2}(\alpha) & = & 0,\\
v_{2,1}(\alpha) & = & \varkappa_1(\alpha)\,\frac{\alpha}{\alpha_0}, & v_{2,2}(\alpha) & = & 0,\\
v_{3,1}(\alpha) & = & \varkappa_2(\alpha), & v_{3,2}(\alpha) & = & 0,\\
v_{4,1}(\alpha) & = & \varkappa_2(\alpha)\,\frac{\alpha-\alpha_0}{1-\alpha_0}, & v_{4,2}(\alpha) & = & 0,\\
v_{5,1}(\alpha) & = & 0, & v_{5,2}(\alpha) & = & \varkappa_1(\alpha),\\
v_{6,1}(\alpha) & = & 0, & v_{6,2}(\alpha) & = & \varkappa_1(\alpha)\,\frac{\alpha_0-\alpha}{\alpha_0},\\
v_{7,1}(\alpha) & = & 0, & v_{7,2}(\alpha) & = & \varkappa_2(\alpha),\\
v_{8,1}(\alpha) & = & 0, & v_{8,2}(\alpha) & = & \varkappa_2(\alpha)\,\frac{1-\alpha}{1-\alpha_0}.
\end{array}$$

By (3)–(4) for each $S = \mathrm{S}(\alpha_0, \mathbf{s}) \in \mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})$, $\mathbf{s} = (s_1, \ldots, s_6)$, we have

$$\begin{aligned}
S \quad = \quad & s_1\mathbf{v}_1 + (s_2 - s_1)\mathbf{v}_2 + s_2\mathbf{v}_3 + (s_3 - s_2)\mathbf{v}_4 \\
& + s_5\mathbf{v}_5 + (s_6 - s_5)\mathbf{v}_6 + s_4\mathbf{v}_7 + (s_5 - s_4)\mathbf{v}_8. \qquad (16)
\end{aligned}$$

For convenience, we will use an alternative notation for S. By setting

$$\begin{aligned}
\delta_1 &:= s_1 - 0,\\
\delta_j &:= s_j - s_{j-1}, \quad j = 2, \ldots, 6,
\end{aligned}$$

where $\delta_1 \in \mathbb{R}$ and $\delta_2, \ldots, \delta_6 \geq 0$, we obtain another representation of S, for which from now on we use the notation $S = \mathrm{S}_d(\alpha_0, \Delta)$, $\Delta = (\delta_1, \ldots, \delta_6)$.

Actually, (16) may be written as

$$\begin{aligned} S &= \delta_1\mathbf{v}_1 + \delta_2\mathbf{v}_2 + (\delta_1+\delta_2)\mathbf{v}_3 + \delta_3\mathbf{v}_4 \\ &\quad +(\delta_1+\delta_2+\delta_3+\delta_4+\delta_5)\mathbf{v}_5 + \delta_6\mathbf{v}_6 \\ &\quad +(\delta_1+\delta_2+\delta_3+\delta_4)\mathbf{v}_7 + \delta_5\mathbf{v}_8 \\ &= \delta_1(\mathbf{v}_1+\mathbf{v}_3+\mathbf{v}_5+\mathbf{v}_7) + \delta_2(\mathbf{v}_2+\mathbf{v}_3+\mathbf{v}_5+\mathbf{v}_7) \\ &\quad +\delta_3(\mathbf{v}_4+\mathbf{v}_5+\mathbf{v}_7) + \delta_4(\mathbf{v}_5+\mathbf{v}_7) + \delta_5(\mathbf{v}_5+\mathbf{v}_8) \\ &\quad +\delta_6\mathbf{v}_6. \end{aligned}$$

We may consider the following vectors

$$\begin{aligned} \mathbf{e}_1 &= \mathbf{v}_1+\mathbf{v}_3+\mathbf{v}_5+\mathbf{v}_7, \\ \mathbf{e}_2 &= \mathbf{v}_2+\mathbf{v}_3+\mathbf{v}_5+\mathbf{v}_7, \\ \mathbf{e}_3 &= \mathbf{v}_4+\mathbf{v}_5+\mathbf{v}_7, \\ \mathbf{e}_4 &= \mathbf{v}_5+\mathbf{v}_7, \\ \mathbf{e}_5 &= \mathbf{v}_5+\mathbf{v}_8, \\ \mathbf{e}_6 &= \mathbf{v}_6, \end{aligned}$$

where $\mathbf{e}_i = (e_{i,1}, e_{i,2}) \in L^2[0,1] \times L^2[0,1]$ for $i \in \{1,\ldots,6\}$. It is easily seen that these vectors are linearly independent. Hence, $\operatorname{span}\{\mathbf{e}_i\}_{i=1,\ldots,6}$ is a linear vector subspace of $L^2[0,1] \times L^2[0,1]$ of dimension 6. It may be shown that using these vectors we can represent the space of α_0-piecewise linear fuzzy numbers with 1 knot as:

$$\mathbb{F}^{\pi(\alpha_0)}(\mathbb{R}) = \left\{ \sum_{i=1}^{6} \delta_i\mathbf{e}_i : \delta_1 \in \mathbb{R},\ \delta_2,\ldots,\delta_6 \in [0,\infty) \right\}.$$

Indeed, by the above reasonings it suffices to verify that for any $\delta_1 \in \mathbb{R}$ and $\delta_2, ..., \delta_6 \in [0,\infty)$ it results that $\sum_{i=1}^{6} \delta_i\mathbf{e}_i \in \mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})$. Since obviously this hold, we get the representation from above.

Now let us denote by $\Phi = \Phi^T = (\phi_{i,j})_{i,j=1,\ldots,6}$, $\phi_{i,j} = \langle \mathbf{e}_i, \mathbf{e}_j \rangle$ (see (6)), the Gram matrix associated with the vectors $\{\mathbf{e}_i\}_{i=1,\ldots,6}$. Since these vectors are linearly independent it follows that Φ is invertible.

In the theorem below we use the following characterization of the best approximation (see e.g. [37, Fact 2.1]). If $(X, \langle\cdot,\cdot\rangle)$ is a Hilbert space, X' is a closed convex subset of X and $x \in X$ then $\bar{x} \in X'$ is the unique best approximation of x relatively to the set X' if and only if $\langle x - \bar{x}, y - \bar{x} \rangle \leq 0$ for any $y \in X'$, where $\overline{x} = P_{X'}(x)$ is the projection of x onto X'.

Theorem 9. *Let $\alpha_0 \in (0,1)$ and $A \in \mathbb{F}(\mathbb{R})$. Then $\Pi_{\alpha_0}(A) = \mathrm{S}_d(\alpha_0, \bar{\Delta})$, $\bar{\Delta} = (\bar{\delta}_1, \ldots, \bar{\delta}_6)$ is the unique best approximation of A relatively to the set $\mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})$ with respect to the metric d if and only if there exists a vector $\bar{\mathbf{z}} = (\bar{z}_1, \ldots, \bar{z}_6)$ such that all the following requirements hold:*

(i) $\bar{\mathbf{z}}^T = \Phi\, \bar{\Delta}^T - \mathbf{b}^T$,

(ii) $\bar{z}_1 = 0$ *and* $\bar{z}_2, \ldots, \bar{z}_6 \geq 0$,

(iii) $\bar{\delta}_i\, \bar{z}_i = 0$ *for all* $i \in \{2, 3, ..., 6\}$,

where $\mathbf{b} = (b_1, \ldots, b_6)$ *such that* $b_i = \langle A, \mathbf{e}_i \rangle$ *for* $i \in \{1, \ldots, 6\}$.

Proof. ($\Longrightarrow$) Let $S^*_{\alpha_0}(A) = \mathrm{S}_d(\alpha_0, \bar{\Delta})$ be the unique best approximation of A. Since $S^*_{\alpha_0}(A) = P_{\mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})}(A)$ then by [37, Fact 2.1] we have

$$\left\langle A - S^*_{\alpha_0}(A), S - S^*_{\alpha_0}(A) \right\rangle \leq 0 \tag{17}$$

for any $S \in \mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})$. Additionally, for all $i \in \{1, \ldots, 6\}$ we get

$$\begin{aligned} \left\langle A - S^*_{\alpha_0}(A), \mathbf{e}_i \right\rangle &= \langle A, \mathbf{e}_i \rangle - \left\langle S^*_{\alpha_0}(A), \mathbf{e}_i \right\rangle \\ &= b_i - \left\langle \sum_{j=1}^{6} \bar{\delta}_j \mathbf{e}_j, \mathbf{e}_i \right\rangle \\ &= b_i - \sum_{j=1}^{6} \bar{\delta}_j \phi_{i,j} \\ &=: -\bar{z}_i. \end{aligned}$$

Please note that for each $S' = S^*_{\alpha_0}(A) + \mathbf{e}_i$ and for some $i \in \{1, \ldots, 6\}$ we have $S' \in \mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})$. Hence, by (17), $\left\langle A - S^*_{\alpha_0}(A), \mathbf{e}_i \right\rangle \leq 0$ or, equivalently, $\bar{z}_i \geq 0$ for all i.

As $S^*_{\alpha_0}(A) - \mathbf{e}_1 \in \mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})$ and $\left\langle A - S^*_{\alpha_0}(A), -\mathbf{e}_1 \right\rangle = -\left\langle A - S^*_{\alpha_0}(A), \mathbf{e}_1 \right\rangle$, we have $\left\langle A - S^*_{\alpha_0}(A), \mathbf{e}_1 \right\rangle = \bar{z}_1 = 0$.

If $\bar{\delta}_i > 0$ for some $i \in \{2, 3, ..., 6\}$ then $S = S^*_{\alpha_0}(A) - \bar{\delta}_i \mathbf{e}_i \in \mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})$. Hence we easily obtain $\left\langle A - S^*_{\alpha_0}(A), \mathbf{e}_i \right\rangle = \bar{z}_i = 0$ for any i such that $\bar{\delta}_i > 0$.

($\Longleftarrow$) Let $S = \mathrm{S}_d(\alpha_0, \bar{\Delta})$ and $\bar{\mathbf{z}}$ be such that the properties (i)–(iii) hold. We will prove that $S^*_{\alpha_0}(A) = S$. Let $M = \{i \in \{2, 3, ..., 6\} : \bar{\delta}_i = 0\}$. Our hypothesis implies that $\langle S - A, \mathbf{e}_i \rangle \leq 0$ for $i \in M$, and $\langle S - A, \mathbf{e}_i \rangle = 0$ whenever $i \notin M$.

Let us take any $S' = \mathrm{S}_d(\alpha_0, \Delta') \in \mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})$, $\Delta' = (\delta'_1, \ldots, \delta'_6)$. Recall that $S' = \sum_{i=1}^{6} \delta'_i \mathbf{e}_i$. We have:

$$\begin{aligned} \langle A - S, S' - S \rangle &= \left\langle A - S, \sum_{i=1}^{6} (\delta'_i - \bar{\delta}_i)\, \mathbf{e}_i \right\rangle \\ &= \sum_{i \in M} (\delta'_i - \bar{\delta}_i) \, \langle A - S, \mathbf{e}_i \rangle \, . \end{aligned}$$

For any $i \in M$ we have $\bar{\delta}_i = 0$ and $\delta'_i \geq 0$. Thus, $\delta'_i - \bar{\delta}_i \geq 0$ so the above sum contains only nonpositive terms. Therefore, since (17) holds, we have $S^*_{\alpha_0}(A) = S$. □

By Theorem 9 we may construct an algorithm for finding the best α_0-piecewise linear approximation of a given fuzzy number A.

4.3. Algorithms

Let us adopt the following notation:

$$\begin{aligned} w_1 &:= \int_0^{\alpha_0} A_L(\beta)\, d\beta, \\ w_2 &:= \frac{1}{\alpha_0} \int_0^{\alpha_0} \beta\, A_L(\beta)\, d\beta, \\ w_3 &:= \int_{\alpha_0}^{1} A_L(\beta)\, d\beta, \\ w_4 &:= \frac{1}{1-\alpha_0} \int_{\alpha_0}^{1} \beta\, A_L(\beta)\, d\beta - \frac{\alpha_0}{1-\alpha_0} w_3, \end{aligned}$$

$$\begin{aligned} w_5 &:= \int_0^{\alpha_0} A_U(\beta)\, d\beta, \\ w_6 &:= w_5 - \frac{1}{\alpha_0} \int_0^{\alpha_0} \beta\, A_U(\beta)\, d\beta, \\ w_7 &:= \int_{\alpha_0}^{1} A_U(\beta)\, d\beta, \\ w_8 &:= \frac{1}{1-\alpha_0} w_7 - \frac{1}{1-\alpha_0} \int_{\alpha_0}^{1} \beta\, A_U(\beta)\, d\beta. \end{aligned}$$

Then

$$\begin{aligned} \mathbf{b} &= (b_1, \ldots, b_6) \qquad (18)\\ &= (w_1 + w_3 + w_5 + w_7, w_2 + w_3 + w_5 + w_7, w_4 + w_5 + w_7, w_5 + w_7, w_5 + w_8, w_6). \end{aligned}$$

Moreover, it may be shown that if $A_L(0) \geq 0$ then $b_1 \geq \cdots \geq b_6 \geq 0$.

Let

$$\Phi = \begin{bmatrix} 2 & \frac{4-\alpha_0}{2} & \frac{3-\alpha_0}{2} & 1 & \frac{\alpha_0+1}{2} & \frac{\alpha_0}{2} \\ \frac{4-\alpha_0}{2} & \frac{6-2\,\alpha_0}{3} & \frac{3-\alpha_0}{2} & 1 & \frac{\alpha_0+1}{2} & \frac{\alpha_0}{2} \\ \frac{3-\alpha_0}{2} & \frac{3-\alpha_0}{2} & \frac{4-\alpha_0}{3} & 1 & \frac{\alpha_0+1}{2} & \frac{\alpha_0}{2} \\ 1 & 1 & 1 & 1 & \frac{\alpha_0+1}{2} & \frac{\alpha_0}{2} \\ \frac{\alpha_0+1}{2} & \frac{\alpha_0+1}{2} & \frac{\alpha_0+1}{2} & \frac{\alpha_0+1}{2} & \frac{2\,\alpha_0+1}{3} & \frac{\alpha_0}{2} \\ \frac{\alpha_0}{2} & \frac{\alpha_0}{2} & \frac{\alpha_0}{2} & \frac{\alpha_0}{2} & \frac{\alpha_0}{2} & \frac{\alpha_0}{3} \end{bmatrix}.$$

be the Gram matrix associated with the vectors $\{\mathbf{e}_i\}_{i=1,\ldots,6}$.

Below we will show that the following algorithm may be used to compute $\Pi_{\alpha_0}(A)$.

Algorithm 1. For given $\alpha_0 \in (0, 1)$ and $A \in \mathbb{F}(\mathbb{R})$:

1. Using the above equations compute Φ and $\mathbf{b}$;
2. For each possible subset K of the set $\{2, \ldots, 6\}$ do:
   - 2.1. Let $\Phi^K := (\phi^K_{i,j})_{i,j=1,\ldots,6}$ be a matrix such that:

$$\phi^K_{i,j} = \begin{cases} \phi_{i,j} & \text{if } j \notin K, \\ 0 & \text{if } j \in K \text{ and } i \neq j, \\ -1 & \text{if } j \in K \text{ and } i = j; \end{cases}$$

   - 2.2. Compute $(\Phi^K)^{-1}$;
   - 2.3. Let $\mathbf{x}^T := (\Phi^K)^{-1}\,\mathbf{b}^T$;
   - 2.4. If $\mathbf{x}$ is such that $x_2, \ldots, x_6 \geq 0$ then:
     - 2.4.1. Let $\bar{\Delta} = (\bar{\delta}_1, \ldots, \bar{\delta}_6)$ such that for $i = 1, \ldots, 6$

$$\bar{\delta}_i = \begin{cases} x_i & \text{for} \quad i \notin K, \\ 0 & \text{for} \quad i \in K; \end{cases}$$

     - 2.4.2. Return $\mathrm{S}_d(\alpha_0, \bar{\Delta})$ as result and stop;

$\blacksquare$

Please note, that for each K an explicit form of the inverse $(\Phi^K)^{-1}$ as a function of α_0 may be provided (see step 2.2).

Proof of correctness. Our problem consists of finding a solution to a system of 6 linear equations with 11 constrained variables given in Theorem 9 (i). We do not know a priori which set $K \subseteq \{2, 3, 4, 5, 6\}$ that implies $\bar{\delta}_i = 0$ for $i \in K$ generates the solution that meet the criteria (ii) and (iii) in Theorem 9. However, Theorem 8 implies that such K always exists so the algorithm stops after at most $2^5 = 32$ iterations.

Note that although the solution is unique, it may exist more than one K which generates the desired solution. For example, it is easily seen that if $i \notin K$ and $x_i = 0$ in Step 2.3 then for $K' = K \cup \{i\}$ we also obtain a proper result, because in this case we have $\bar{\delta}_i = \bar{z}_i = 0$.

For any $K \subseteq \{2, \ldots, 6\}$ the j-th equation in Theorem 9, $j = 1, \ldots, 6$, may be written as

$$\left\{ \begin{array}{ccc} \bar{z}_1 \mathbf{1}_K(1) & = & \sum_{i=1}^{6} \phi_{1,i}\, \bar{\delta}_i (1 - \mathbf{1}_K(i)) - b_1, \\ \vdots & \vdots & \vdots \\ \bar{z}_6 \mathbf{1}_K(6) & = & \sum_{i=1}^{6} \phi_{6,i}\, \bar{\delta}_i (1 - \mathbf{1}_K(i)) - b_6, \end{array} \right.$$

where $\mathbf{1}$ denotes the indicator function, i.e. $\mathbf{1}_K(i) = 1$ if $i \in K$ and $\mathbf{1}_K(i) = 0$ otherwise. It gives a system of 6 linear equations with 6 variables. Its solution $\mathbf{x} \in \mathbb{R}^6$ may be determined by calculating $\mathbf{x}^T := (\Phi^K)^{-1}\, \mathbf{b}^T$, where the nonsingular matrix (Φ^K) is given in Step 2.1, while $\bar{\delta}_j = x_j$ and $\bar{z}_i = x_i$ for $i \in K$ and $j \notin K$. It is sufficient that the solution fulfills the condition given in Step 2.4. In such case in Step 2.4.2 we obtain $\Pi_{\alpha_0}(A)$ and the proof is complete. $\square$

Please observe that the resulting piecewise linear fuzzy number may easily be expressed in the "standard" form. We have $\mathrm{S}_d(\alpha_0, \mathbf{x}) = \mathrm{S}(\alpha_0, \texttt{cumsum}(\mathbf{x}))$, where $\texttt{cumsum}(x_1, \ldots, x_6) = (x_1, x_1 + x_2, \ldots, x_1 + \cdots + x_6)$ denotes the cumulative sum of $\mathbf{x}$.

Example 1. Consider $A \in \mathbb{F}(\mathbb{R})$ defined as

$$\begin{aligned} A_L(\alpha) &= \begin{cases} 0 & \text{for } \alpha < 0.5, \\ 0.5 & \text{for } \alpha \geq 0.5, \end{cases} \\ A_U(\alpha) &= 1. \end{aligned}$$

Assume that $\alpha_0 = 1/4$. We have $w_1 = 0$, $w_2 = 0$, $w_3 = 1/4$, $w_4 = 1/6$, $w_5 = 1/4$, $w_6 = 1/8$, $w_7 = 3/4$, and $w_8 = 3/8$. Thus, $\mathbf{b} = (5/4, 5/4, 7/6, 1, 5/8, 1/8)$. Moreover for $K = \emptyset$ we have:

$$(\Phi^{\emptyset})^{-1} = \begin{bmatrix} 13 & -15 & 3 & -1 & 0 & 0 \\ -15 & 21 & -9 & 3 & 0 & 0 \\ 3 & -9 & 13 & -7 & 0 & 0 \\ -1 & 3 & -7 & 10 & -7 & 3 \\ 0 & 0 & 0 & -7 & 13 & -9 \\ 0 & 0 & 0 & 3 & -9 & 21 \end{bmatrix}$$

we compute $\mathbf{x}^T := (\Phi^{\emptyset})^{-1}\,\mathbf{b}^T = (0, 0, 2/3, 1/3, 0, 0)^T$. As $x_2, \ldots, x_6 \geq 0$, the resulting nearest approximation of A is $\Pi_{1/4}(A) = \mathrm{S}_d(1/4, \mathbf{x}) = \mathrm{S}(1/4, (0, 0, 2/3, 1, 1, 1))$. ■

Example 2. Let $\alpha_0 = 1/4$ and $A' \in \mathbb{F}(\mathbb{R})$ such that

$$\begin{aligned} A'_L(\alpha) &= 0, \\ A'_U(\alpha) &= (1-\alpha)^2. \end{aligned}$$

We have $w_1 = w_2 = w_3 = w_4 = 0$, $w_5 = 37/192$, $w_6 = 81/768$, $w_7 = 27/192$, $w_8 = 81/768$, $\mathbf{b} = (1/3, 1/3, 1/3, 1/3, 229/768, 81/768)$. Let us try $K = \emptyset$. We have $\mathbf{x}^T := (\Phi^{\emptyset})^{-1}\,\mathbf{b}^T = (0, 0, 0, -80/768, 456/768, 408/768)^T$ and we note that $x_4 < 0$. Therefore, we have to try a different set K.

For $K' = \{2, 3, 4\}$ we have

$$(\Phi^{K'})^{-1} = \begin{bmatrix} \frac{5}{6} & 0 & 0 & 0 & -\frac{7}{6} & \frac{1}{2} \\ \frac{43}{48} & -1 & 0 & 0 & \frac{7}{48} & -\frac{1}{16} \\ \frac{23}{48} & 0 & -1 & 0 & \frac{35}{48} & -\frac{5}{16} \\ \frac{1}{6} & 0 & 0 & -1 & \frac{7}{6} & -\frac{1}{2} \\ -\frac{7}{6} & 0 & 0 & 0 & \frac{29}{6} & -\frac{11}{2} \\ \frac{1}{2} & 0 & 0 & 0 & -\frac{11}{2} & \frac{39}{2} \end{bmatrix}$$

which gives $\mathbf{x}'^T := (\Phi^{K'})^{-1}\,\mathbf{b}^T = (-5/288, 5/2304, 25/2304, 5/288, 17/36, 7/12)^T$. As $x'_2, \ldots, x'_6 \geq 0$, we set $x'_i := 0$ for $i \in K'$ and the resulting nearest approximation of A' is $\Pi_{1/4}(A) = \mathrm{S}(1/4, (-5/288, -5/288, -5/288, -5/288, 131/288, 299/288))$. ■

We see that the above algorithm requires up to 32 steps. Although a computer implementation of such algorithm is very fast, it would be better to propose a method to select K in a less naïve way. Please note that our aim is to find $(\bar{\Delta}, \bar{\mathbf{z}})$ such that

$$\bar{\Delta}^T = \Phi^{-1}(\mathbf{b}^T + \bar{\mathbf{z}}^T)$$

and which fulfill the conditions (ii)–(iii) in Theorem 9. Equivalently, for $\alpha_0 \in (0,1)$ we find $(\bar{\Delta}, \bar{\mathbf{z}})$ such that

$$\begin{bmatrix} \bar{\delta}_1 \\ \bar{\delta}_2 \\ \bar{\delta}_3 \\ \bar{\delta}_4 \\ \bar{\delta}_5 \\ \bar{\delta}_6 \end{bmatrix} = \begin{bmatrix} \frac{\alpha_0+3}{\alpha_0} & -\frac{3\,\alpha_0+3}{\alpha_0} & 3 & -1 & 0 & 0 \\ -\frac{3\,\alpha_0+3}{\alpha_0} & \frac{9\,\alpha_0+3}{\alpha_0} & -9 & 3 & 0 & 0 \\ 3 & -9 & \frac{9\,\alpha_0-12}{\alpha_0-1} & \frac{6-3\,\alpha_0}{\alpha_0-1} & 0 & 0 \\ -1 & 3 & \frac{6-3\,\alpha_0}{\alpha_0-1} & \frac{2\,\alpha_0-8}{\alpha_0-1} & \frac{6-3\,\alpha_0}{\alpha_0-1} & 3 \\ 0 & 0 & 0 & \frac{6-3\,\alpha_0}{\alpha_0-1} & \frac{9\,\alpha_0-12}{\alpha_0-1} & -9 \\ 0 & 0 & 0 & 3 & -9 & \frac{9\,\alpha_0+3}{\alpha_0} \end{bmatrix} \begin{bmatrix} b_1 + 0 \\ b_2 + \bar{z}_2 \\ b_3 + \bar{z}_3 \\ b_4 + \bar{z}_4 \\ b_5 + \bar{z}_5 \\ b_6 + \bar{z}_6 \end{bmatrix}.$$

and $\bar{\delta}_i \wedge \bar{z}_i = 0$ for $i \in \{2, \ldots, 6\}$.

Observe that $\phi^{-1}_{i,i} \geq 0$ and $\arg\max_j |\phi^{-1}_{j,i}| = i$ for all $\alpha_0 \in (0,1)$ and any $i \geq 2$, i.e. the element with the greatest absolute value in each column of Φ^{-1} lies on the diagonal and is always positive.

Let us fix $\mathbf{b} \in \mathbb{R}^6$. Consider any $\hat{\Delta} \in \mathbb{R}^6$ and $\hat{\mathbf{z}} \in \mathbb{R}^6_{0+}$ such that $\hat{\Delta}^T = \Phi^{-1}(\mathbf{b}^T + \hat{\mathbf{z}}^T)$ and $\hat{z}_1 = 0$. By setting $\hat{z}'_i := \hat{z}_i + \zeta_i$ for any $i \in \{2, \ldots, 6\}$ and $\zeta_i > 0$ we get $\hat{\Delta}' := \hat{\Delta} + (\zeta_i\,\phi^{-1}_{1,i}, \ldots, \zeta_i\,\phi^{-1}_{6,i})$ with $\hat{\delta}'_i > \hat{\delta}_i$.

Moreover, for any $\hat{K} \subseteq \{2, \ldots, 6\}$ there always exists a vector $Z = (0, \zeta_2, \ldots, \zeta_6)$ with $\zeta_l = 0$, $l \notin \hat{K}$, for which by setting $\hat{\mathbf{z}}' := \hat{\mathbf{z}} + Z$ we obtain $\hat{\Delta}'$ such that $\hat{\delta}'_k = 0$ for $k \in \hat{K}$. However, it is possible that $\hat{z}'_i < 0$ for some i.

For any vector $\mathbf{x} \in \mathbb{R}^6$ and any index set K let $\mathbf{x}|_K$ denote the subvector formed by elements x_i with $i \in K$, i.e. $\mathbf{x}|_K = (x_{k_1}, \ldots, x_{k_{|K|}})$ such that $k_1, \ldots, k_K \in K$ and $k_1 < \cdots < k_{|K|}$. A similar subsetting operation may be introduced on matrices.

Basing on the above-presented discussion, we propose the following iterative algorithm which possibly gives the desired solution in shorter time than the previous one.

Algorithm 2. For given $\alpha_0 \in (0,1)$ and $A \in \mathbb{F}(\mathbb{R})$:

1. Using the above equations compute Φ^{-1} and $\mathbf{b}$;
2. Let $\hat{\mathbf{z}} := (0,0,0,0,0,0)$;
3. Let $\hat{K} := \emptyset$;

4. Let $\hat{\Delta}^T := \Phi^{-1}\,\mathbf{b}^T$;
5. Let $m := \arg\min_{m=2,\dots,6} \hat{\delta}_m$;
6. While $\hat{\delta}_m < 0$ do:
   6.1. $\hat{K} := \hat{K} \cup \{m\}$;
   6.2. Solve $\Phi^{-1}|_{\hat{K},\hat{K}}\, Z^T|_{\hat{K}} = -\hat{\Delta}^T|_{\hat{K}}$ for $Z^T|_{\hat{K}}$;
   6.3. Set $\hat{z}_k := \hat{z}_k + \zeta_k$ for $k \in \hat{K}$;
   6.4. $\hat{\Delta}^T := \Phi^{-1}\,(\mathbf{b}^T + \hat{\mathbf{z}}^T)$;
   6.5. $m := \arg\min_{m=2,\dots,6} \hat{\delta}_m$;
7. Return $\mathrm{S}_d(\alpha_0, \hat{\Delta})$ as result;

■

Proof of correctness. Note that in Step 6.2 the submatrix $\Phi^{-1}|_{\hat{K},\hat{K}}$ is always invertible. It is used to obtain $\hat{\mathbf{z}}$ in Step 6.3 for which we get $\hat{\Delta}|_{\hat{K}} = 0$ in Step 6.4. It is easily seen that the algorithm finds a solution after performing at most 6 iterations, because after each execution of the while-loop we are sure that $\hat{\Delta}|_{\hat{K}} = 0$ and always a new element is added to $\hat{K} \subseteq \{2, \dots, 6\}$.

If we show that in each iteration we have $\hat{\mathbf{z}} \geq 0$ then we will be able to conclude that this procedure solves our problem. Let us analyze then the elements of Φ^{-1}. We see that if $\hat{\delta}_3 < 0$ then the only possibility to obtain $\hat{\delta}_3' = 0$ is to set $\zeta_3 > 0$. Note also that $\zeta_3 > 0$ implies that $\hat{\delta}_2' < \hat{\delta}_2$ and $\hat{\delta}_4' < \hat{\delta}_4$. Moreover, $\zeta_2 > 0$ or $\zeta_4 > 0$ implies that $\hat{\delta}_3' < \hat{\delta}_3$. Thus, setting $3 \in \hat{K}$ whenever $\hat{\delta}_3 < 0$ (in any iteration) always implies that $\hat{z}_3 \geq 0$.

If $\hat{\delta}_2 < 0$ we may set $\hat{\delta}_2' = 0$ either by choosing $\zeta_2 > 0$ or $\zeta_4 > 0$ because $\phi^{-1}_{2,2} > 0$ and $\phi^{-1}_{2,4} > 0$. A similar remark holds for $\hat{\delta}_4 < 0$. By choosing to set ζ_m where $m = \arg\min_{m\in\{2,4\}} \hat{\delta}_m$ we ensure that if $\hat{\delta}_{6-m}' < 0$ still holds and we have to set $\zeta_{6-m}' > 0$ (which implies that $\zeta_m' < 0$) then we will anyway obtain $\hat{z}_2'' \geq 0$ and $\hat{z}_4'' \geq 0$, because $\zeta_m > -\zeta_m'$.

As there is a kind of symmetry between the behavior of $\hat{\delta}_2, \hat{\delta}_3, \hat{\delta}_4$ and $\hat{\delta}_6, \hat{\delta}_5, \hat{\delta}_4$, respectively, we get the desired conclusion. □

Example 3. The above algorithm has been implemented in the `FuzzyNumbers` package [16] for `R` [28]. Let us consider the fuzzy number A' from the previous example and now determine its best $\alpha_0 = 1/4$-piecewise linear approximation numerically.

```
# define A' A <- FuzzyNumber(0,0,0,1,
        lower=function(a) a, # any function [0,1]->[0,1]
```

```
        upper=function(a) (1-a)^2) # function [0,1]->[1,0]

# calculate best piecewise linear approximation numerically
S <- piecewiseLinearApproximation(A, verbose=TRUE,
   method="NearestEuclidean", knot.n=1, knot.alpha=0.25)
## Pass 1: K={   }, d=(     0,      0,      0, -0.10,  0.59, 0.53)
##                  z=(     0,      0,      0,     0,     0, 0   )
## Pass 2: K={  4}, d=(-0.010,  0.031, -0.073,     0,  0.52, 0.56)
##                  z=(     0,      0,      0,  0.01,     0, 0   )
## Pass 3: K={ 34}, d=( 0.010, -0.031,      0,     0,  0.48, 0.58)
##                  z=(     0,      0,  0.009, 0.017,     0, 0   )
## Pass 4: K={234}, d=(-0.017,      0,      0,     0,  0.47, 0.58)
##                  z=(     0,  0.002,  0.011, 0.017,     0,    0)
## DONE in 4 iterations.

# print result (alpha-cuts):
print(S["allknots"])
##         alpha        left       right
## supp     0.00 -0.01736111  1.03819444
## knot_1   0.25 -0.01736111  0.45486111
## core     1.00 -0.01736111 -0.01736111
```

We see that the algorithm converged in 4 iterations. ■

4.4. Properties of the approximation operator

When discussing properties of the approximation operator producing outputs closest to the original fuzzy number is quite important. However, a good approximation operator should possess some other desirable properties. A broad list of criteria which the approximation operator should or just can possess is given in [22]. Below we examine the properties of the operator Π_{α_0}.

Theorem 10. *For any $\alpha_0 \in [0,1]$ the Π_{α_0} approximation operator fulfills the following properties.*

(i) *identity, i.e.* $\Pi_{\alpha_0}(A) = A \quad (\forall A \in \mathbb{F}^{\pi(\alpha_0)}(\mathbb{R}))$;

(ii) *invariance to translation, i.e.* $\Pi_{\alpha_0}(A + z) = \Pi_{\alpha_0}(A) + z \quad (\forall A \in \mathbb{F}(\mathbb{R})) \ (\forall z \in \mathbb{R})$;

(iii) *scale invariance, i.e.* $\Pi_{\alpha_0}(\lambda \cdot A) = \lambda \cdot \Pi_{\alpha_0}(A) \quad (\forall A \in \mathbb{F}(\mathbb{R}))\ (\forall \lambda \in \mathbb{R})$;

(iv) *Lipschitz-continuity, i.e.* $d(\Pi_{\alpha_0}(A), \Pi_{\alpha_0}(B)) \le d(A, B) \quad (\forall A, B \in \mathbb{F}(\mathbb{R}))$.

Proof. (i) Trivial.

(ii) Let $z \in \mathbb{R}$ be arbitrary. It is clear that $d(A+z, \Pi_{\alpha_0}(A)+z) = d(A, \Pi_{\alpha_0}(A))$. For any $S \in \mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})$ we have $S - z \in \mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})$ and hence

$$d(A+z, S) = d(A, S-z) \geq d(A, \Pi_{\alpha_0}(A)) = d(A+z, \Pi_{\alpha_0}(A)+z).$$

Since $\Pi_{\alpha_0}(A) + z \in \mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})$, by the uniqueness of $\Pi_{\alpha_0}(A+z)$ on $\mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})$, we get $\Pi_{\alpha_0}(A+z) = \Pi_{\alpha_0}(A) + z$, which means that the operator Π_{α_0} is invariant to translations.

(iii) Take $\lambda \in \mathbb{R}$, $\lambda \neq 0$. It holds $d(\lambda \cdot A, \lambda \cdot \Pi_{\alpha_0}(A)) = |\lambda|\, d(A, \Pi_{\alpha_0}(A))$. For any $S \in \mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})$ we have $\frac{1}{\lambda} \cdot S \in \mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})$ and hence

$$\begin{aligned} d(\lambda A, S) &= |\lambda|\, d(A, \tfrac{1}{\lambda} \cdot S) \\ &\geq |\lambda|\, d(A, \Pi_{\alpha_0}(A)) \\ &= |\lambda|\, d(\tfrac{1}{\lambda} \cdot (\lambda \cdot A), \tfrac{1}{\lambda} \cdot (\lambda \cdot \Pi_{\alpha_0}(A))) \\ &= |\lambda| \left|\tfrac{1}{\lambda}\right| d(\lambda \cdot A, \lambda \cdot \Pi_{\alpha_0}(A)) \\ &= d(\lambda \cdot A, \lambda \cdot \Pi_{\alpha_0}(A)). \end{aligned}$$

Since $\lambda \cdot \Pi_{\alpha_0}(A) \in \mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})$, by the uniqueness of $\Pi_{\alpha_0}(\lambda \cdot A)$ on $\mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})$, we get $\Pi_{\alpha_0}(\lambda \cdot A) = \lambda \cdot \Pi_{\alpha_0}(A)$, which means that the operator Π_{α_0} is scale invariant on $\mathbb{R} \setminus \{0\}$.

On the other hand, it is easily seen that for $\lambda = 0$ we have $\Pi_{\alpha_0}(\lambda \cdot A) = \lambda \cdot \Pi_{\alpha_0}(A) = O$, where O is the real number 0 represented as a fuzzy number. We may thus conclude that Π_{α_0} is scale invariant on $\mathbb{R}$.

(iv) As we have already noticed, for a given fuzzy number A, $\Pi_{\alpha_0}(A)$ is the projection of A with respect to the metric $\widetilde{d}$ onto the closed convex set $\mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})$ (which is a subset of $L^2[0,1] \times L^2[0,1]$). Actually, it means that Π_{α_0} is a projector to a closed convex subset of a Hilbert space. From the Hilbert space theory (see e.g. [34], Fact 6.4) it is known that such a projector is a non-expansive function. This implies that $\widetilde{d}(\Pi_{\alpha_0}(A), \Pi_{\alpha_0}(B)) \leq \widetilde{d}(A, B)$ for any $A, B \in \mathbb{F}(\mathbb{R})$ and by the properties of metric $\widetilde{d}$ we get $d(\Pi_{\alpha_0}(A), \Pi_{\alpha_0}(B)) \leq d(A, B)$ for any $A, B \in \mathbb{F}(\mathbb{R})$. Therefore, the operator Π_{α_0} satisfies the Lipschitz condition (which also implies its continuity). $\square$

5. Finding a knot α_0 for the best piecewise linear approximation

5.1. The problem

Until this moment we considered situations when the α_0-cut representing the knot of the piecewise linear 1-knot fuzzy number approximating given fuzzy number A was fixed in advance, i.e. we were looking for the solution $S(A)$ in $\mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})$. However, now we will discuss a more general problem: we will try to find the best piecewise linear 1-knot fuzzy number approximation of a given fuzzy number A without pre-setting any α_0-cut as a knot. In other words our goal now is to indicate the optimal knot for the best piecewise linear 1-knot fuzzy number approximation.

More formally, we are looking for such $S^* \in \mathbb{F}^{\pi[0,1]}(\mathbb{R})$ that

$$d(A, S^*) = \min_{S \in \mathbb{F}^{\pi[0,1]}(\mathbb{R})} d(A, S),$$

where A is any fuzzy number and d denotes, as before, metric (2).

It is worth stressing that the problem is much more difficult now, because $\mathbb{F}^{\pi[0,1]}(\mathbb{R})$ is not a convex set and for this reason we cannot use Hilbert space theory as we did for the space $\mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})$. We show that there is a solution for any $A \in \mathbb{F}(\mathbb{R})$ but is not necessarily unique.

5.2. Existence

Let us define $D_A : [0,1] \to \mathbb{R}_{0+}$ such that $D_A(\alpha) = d(A, \Pi_\alpha(A))$. It appears that our interest in this section is to find a global minimum (possibly non-unique) of D_A. Let us first note two important facts.

Proposition 11. *For any $A \in \mathbb{F}(\mathbb{R})$ the following properties hold:*

(i) D_A *is continuous on* $[0,1]$,
(ii) $0 \le D_A(\alpha) \le D_A(0) = D_A(1) \quad (\forall \alpha \in [0,1])$.

Proof. (i) We will prove the continuity of $D_A(\alpha)$ separately at (a) $\alpha \in (0,1)$, (b) $\alpha = 0$, and (c) $\alpha = 1$. It is so because cases $\alpha \in \{0,1\}$ cannot be computed with the algorithms developed in the previous section.

(a) Assume that $\alpha \in (0,1)$. Let $\Pi_\alpha(A) = \mathrm{S}_d(\alpha, \bar{\Delta}_\alpha)$. Clearly, the elements of $\bar{\Delta}_\alpha = (\bar{\delta}_1(\alpha), \ldots, \bar{\delta}_6(\alpha))$ may be uniquely determined for all $\alpha \in (0,1)$. It suffices to show that each $\bar{\delta}_i$ is a continuous function. This is because if $(\forall i \in \{1, \ldots, 6\})$ $(\forall \alpha_0 \in (0,1))$ $\lim_{\alpha \to \alpha_0} \bar{\delta}_i(\alpha) = \bar{\delta}_i(\alpha_0)$ then, by Lemma 3, Proposition 4, and the triangle inequality:

$$\left| d\left(A, \mathrm{S}_d(\alpha, \bar{\Delta}_\alpha)\right) - d\left(A, \mathrm{S}_d(\alpha_0, \bar{\Delta}_{\alpha_0})\right) \right| \le d\left(\mathrm{S}_d(\alpha, \bar{\Delta}_\alpha), \mathrm{S}_d(\alpha_0, \bar{\Delta}_{\alpha_0})\right) \to 0,$$

which implies the continuity of D_A.

Let $\mathbf{b}_\alpha(A) = \mathbf{b}_\alpha = (b_1(\alpha), \ldots, b_6(\alpha))$ be calculated for a fixed A and each α using (18). Please note that $\mathbf{b}_\alpha$ contains all the information on the fuzzy number A that is needed for its approximation and that b_1, b_4 are constant functions. What is important, even if A_L or A_U are not continuous they are still monotonic and integrable on [0,1], therefore for any i the function b_i is continuous (as it is a sum of continuous functions). Moreover, let $\Phi_\alpha^K = (\phi_{i,j}^K(\alpha))_{i,j=1,\ldots,6}$ denote the matrix defined in Step 2.1 of Algorithm 1 (for any K). It may be checked that for each $K \subseteq \{2, \ldots, 6\}$, $(\Phi_\alpha^K)^{-1}$ exists for any α. Additionally, $\phi_{i,j}^K$ and $(\phi^K)_{i,j}^{-1}$ are continuous functions for all i, j.

Thus we conclude that for all K if $\mathbf{x}_\alpha^K = (x_1^K(\alpha), \ldots, x_6^K(\alpha))$ is such that $(\mathbf{x}_\alpha^K)^T := (\Phi_\alpha^K)^{-1}\, \mathbf{b}_\alpha^T$ (see Step 2.3 of Algorithm 1) then the function x_i^K is continuous for any i, because it is obtained by adding and multiplying continuous functions.
In order to prove that each $\overline{\delta}_i$ is continuous on $(0,1)$ it suffices to show that for any $a, b \in (0,1)$ with $a < b$, we have the continuity of each $\overline{\delta}_i$ on $[a,b]$. So, let us fix $a, b \in (0,1)$ with $a < b$. For any $K \subseteq \{2, ..., 6\}$ let us define the following set

$$\mathcal{I}(a, b, K) = \{\alpha \in [a, b] : \ \Pi_\alpha(A) \text{ is generated by } K\}.$$

Let us denote with $\mathcal{P}$ the power set of $\{2, ..., 6\}$. It is immediately seen that

$$[a, b] = \bigcup_{K \in \mathcal{P}} \mathcal{I}(a, b, K).$$

Next, we prove that each $\mathcal{I}(a, b, K)$ is a closed subset of $[a, b]$ (we exclude those sets that are empty since they do not affect at all the proof). To this end, let $(\alpha_n)_{n \geq 1}$ be a sequence in $\mathcal{I}(a, b, K)$ so that $\alpha_n \to \alpha_0$. We have to prove that $\alpha_0 \in \mathcal{I}(a, b, K)$. We observe that by step 2.4 in Algorithm 1 we have $x_i(\alpha_n) \geq 0$ for all $i \in \{2, ..., 6\}$. By passing to limit as $n \to \infty$ and by continuity of each x_i, we get $x_i(\alpha_0) \geq 0$ for any $i \in \{2, ..., 6\}$. Therefore, according to step 2.4.1 in Algorithm 1, it results that $\Pi_{\alpha_0}(A)$ is generated by K too. This means that $\alpha_0 \in \mathcal{I}(a, b, K)$ and hence $\mathcal{I}(a, b, K)$ is a closed subset of $[a, b]$. Please note that for α_0 Algorithm 1 is applicable since by our assumptions we have $0 < a \leq \alpha_0 \leq b < 1$. Since $\mathcal{I}(a, b, K)$ is closed, it is immediate that for any sequence $(\alpha_n)_{n \geq 1}$ in $\mathcal{I}(a, b, K)$ so that $\alpha_n \to \alpha_0$ we have $\overline{\delta}_i(\alpha_n) \to \overline{\delta}_i(\alpha_0)$, $i \in \{1, ..., 6\}$. This is because the restriction of $\overline{\delta}_i$ to $\mathcal{I}(a, b, K)$ is x_i if $i \notin K$ and the null function otherwise.

Now, we prove the continuity of each $\overline{\delta}_i$, $i \in \{1, ..., 6\}$ on $[a, b]$ by using the Heine criterion. For this reason let $(\alpha_n)_{n \geq 1}$ be a sequence in $[a, b]$ so that $\alpha_n \to \alpha_0$. We have to prove that $\overline{\delta}_i(\alpha_n) \to \overline{\delta}_i(\alpha_0)$ for any $i \in \{2, ..., 6\}$. For this sequence there exist the sets $K_{l_j} \subseteq \{1, ..., 6\}$, $j \in \{1, ..., r\}$ (here r is at most 32, the number of elements of $\mathcal{P}$) such that for each $\mathcal{I}(a, b, K_{l_j})$ it holds that the set $\mathbb{N}_j = \{n \in \mathbb{N} : \alpha_n \in \mathcal{I}(a, b, K_{l_j})\}$ is infinite while, for any $K \in \mathcal{P} \setminus \{K_{l_j} : j = 1, \ldots, r\}$ (of course it may happen that the difference is the empty set), the set $\{n \in \mathbb{N} : \alpha_n \in \mathcal{I}(a, b, K)\}$ is empty or finite. Excluding a finite number of terms if necessary (which do not affect at all our reasoning), we can write the sequence $(\alpha_n)_{n\geq 1}$ as the union of the subsequences $(\alpha_{kj_n})_{n\geq 1}$, $j \in \{1, ..., r\}$ where $kj : \mathbb{N} \to \mathbb{N}_j$, $kj(n) = kj_n$ is a strictly increasing bijection. Then it is well known that $\alpha_{kj_n} \to \alpha_0$ for any $j \in \{1, ..., r\}$, which firstly implies that $\alpha_0 \in \mathcal{I}(a, b, K_{l_j})$ for any $j \in \{1, ..., r\}$ and then recalling the property mentioned a few lines above, it results that $\overline{\delta}_i(\alpha_{kj_n}) \to \overline{\delta}_i(\alpha_0)$, $i \in \{1, ..., 6\}$. This implies that for any $i \in \{1, ..., 6\}$ the sequence $\left(\overline{\delta}_i(\alpha_n)\right)_{n\geq 1}$ is the union of the subsequences (excluding possibly a finite number of terms)$\left(\overline{\delta}_i(\alpha_{kj_n})\right)_{n\geq 1}$, $j \in \{1, ..., r\}$ and each one of these sequences converges to $\overline{\delta}_i(\alpha_0)$. This means nothing else but the fact that for any $i \in \{1, ..., 6\}$ we have $\lim_{n\to\infty} \overline{\delta}_i(\alpha_n) = \overline{\delta}_i(\alpha_0)$. Thus, by the Heine criterion we get that for any $i \in \{1, ..., 6\}$, $\overline{\delta}_i$ is continuous on $[a, b]$. From here, as we have already discussed that, it follows that the function D_A is continuous on $(0, 1)$.

(b) Assume that $\alpha = 0$. Let $L_1 = \int_0^1 A_L(\beta)\, d\beta$, $L_2 = \int_0^1 \beta\, A_L(\beta)\, d\beta$, $U_1 = \int_0^1 A_U(\beta)\, d\beta$, and $U_2 = \int_0^1 \beta\, A_U(\beta)\, d\beta$. We have $L_1 \leq U_1$ and $L_2 \leq U_2$. As the approximation operator is translation invariant we may assume without loss of generality that $A_L(0) \geq 0$. This implies $L_2 \leq L_1 \leq 2\, L_2$, and $2\, U_2 \leq U_1$.

It may be seen that

$$\mathbf{b}_\beta \overset{\beta\to 0}{\to} (U_1 + L_1, U_1 + L_1, U_1 + L_2, U_1, U_1 - U_2, 0).$$

Note it holds $b_1(0) = b_2(0)$.

Let us set $\bar{\delta}'_1 := \bar{\delta}_1 + \bar{\delta}_2$. Equation (i) in Theorem 9 may be written as

$$\bar{\mathbf{z}}^T_\beta = \Phi'_\beta \begin{bmatrix} \bar{\delta}'_1(\beta) \\ \bar{\delta}_2(\beta) \\ \bar{\delta}_3(\beta) \\ \bar{\delta}_4(\beta) \\ \bar{\delta}_5(\beta) \\ \bar{\delta}_6(\beta) \end{bmatrix} + \mathbf{b}^T_\beta$$

where

$$\Phi'_\beta = \begin{bmatrix} 2 & -\frac{\beta}{2} & \frac{3-\beta}{2} & 1 & \frac{\beta+1}{2} & \frac{\beta}{2} \\ \frac{4-\beta}{2} & -\frac{\beta}{6} & \frac{3-\beta}{2} & 1 & \frac{\beta+1}{2} & \frac{\beta}{2} \\ \frac{3-\beta}{2} & 0 & \frac{4-\beta}{3} & 1 & \frac{\beta+1}{2} & \frac{\beta}{2} \\ 1 & 0 & 1 & 1 & \frac{\beta+1}{2} & \frac{\beta}{2} \\ \frac{\beta+1}{2} & 0 & \frac{\beta+1}{2} & \frac{\beta+1}{2} & \frac{2\beta+1}{3} & \frac{\beta}{2} \\ \frac{\beta}{2} & 0 & \frac{\beta}{2} & \frac{\beta}{2} & \frac{\beta}{2} & \frac{\beta}{3} \end{bmatrix} \tag{19}$$

$$\stackrel{\beta\to 0}{\to} \begin{bmatrix} 2 & 0 & \frac{3}{2} & 1 & \frac{1}{2} & 0 \\ 2 & 0 & \frac{3}{2} & 1 & \frac{1}{2} & 0 \\ \frac{3}{2} & 0 & \frac{4}{3} & 1 & \frac{1}{2} & 0 \\ 1 & 0 & 1 & 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{20}$$

Clearly, Φ_0 is singular. However, as D_A is continuous on $(0,1)$, and for $\alpha = 0$ we have $\mathbb{F}^{\pi(\alpha)}(\mathbb{R}) = \mathbb{F}^T(\mathbb{R})$, it suffices to show that $\bar{\delta}_1(\beta)' \to t_1$, $\bar{\delta}_1(\beta)' + \bar{\delta}_3(\beta) \to t_2$, $\bar{\delta}_1(\beta)' + \bar{\delta}_3(\beta) + \bar{\delta}_4(\beta) \to t_3$, and $\bar{\delta}_1(\beta)' + \bar{\delta}_3(\beta) + \bar{\delta}_4(\beta) + \bar{\delta}_5(\beta) \to t_4$ as $\beta \to 0$, where t_1, t_2, t_3, t_4 are such that $\mathrm{T}(t_1, t_2, t_3, t_4)$ is the nearest trapezoidal approximation of A given in e.g. [5]. Note we may set $\delta_2(0) = 0$ and $\delta_6(0) = 0$ with no loss in generality as their values do not affect $D_A(0)$.

Equivalently, we should show that the solution to the problem

$$\begin{bmatrix} 0 \\ z_3(0) \\ z_4(0) \\ z_5(0) \end{bmatrix} = \begin{bmatrix} 2 & \frac{3}{2} & 1 & \frac{1}{2} \\ \frac{3}{2} & \frac{4}{3} & 1 & \frac{1}{2} \\ 1 & 1 & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \delta'_1(0) \\ \delta_3(0) \\ \delta_4(0) \\ \delta_5(0) \end{bmatrix} + \begin{bmatrix} U_1 + L_1 \\ U_1 + L_2 \\ U_1 \\ U_1 - U_2 \end{bmatrix}$$

written for brevity as

$$\hat{\mathbf{z}}^T = \hat{\Phi}\hat{\Delta}^T + \hat{\mathbf{b}}^T$$

with constraints $\hat{z}_3(0) \wedge \hat{\delta}_3(0) = 0, \ldots, \hat{z}_5(0) \wedge \hat{\delta}_5(0) = 0$ generates a fuzzy number equal to $\mathrm{T}(t_1, t_2, t_3, t_4)$.

Let us consider all possible $K \subseteq \{2, 3, 4\}$.

1. $K = \emptyset$. We have:

$$(\hat{\Phi}^K)^{-1} = \begin{bmatrix} 4 & -6 & 2 & 0 \\ -6 & 12 & -6 & 0 \\ 2 & -6 & 8 & -6 \\ 0 & 0 & -6 & 12 \end{bmatrix}$$

Please note that as $12\,L_2 \geq 6\,L_1$ and $12\,U_2 \leq 6\,U_1$, then we always have $\hat{\delta}_3 \geq 0$ and $\hat{\delta}_5 \geq 0$. Thus, we need not consider $K = \{2\}$, $K = \{4\}$, and $K = \{2,4\}$. The solution to this case is equivalent to case (i) in [5, Corollary 8].

2. $K = \{3\}$.

$$(\hat{\Phi}^K)^{-1} = \begin{bmatrix} 7/2 & -9/2 & 0 & 3/2 \\ -9/2 & 15/2 & 0 & -9/2 \\ -1/4 & 3/4 & -1 & 3/4 \\ 3/2 & -9/2 & 0 & 15/2 \end{bmatrix}$$

The solution to this case is equivalent to case (ii) in [5, Corollary 8].

3. $K = \{3,4\}$.

$$(\hat{\Phi}^K)^{-1} = \begin{bmatrix} 16/5 & -18/5 & 0 & 0 \\ -18/5 & 24/5 & 0 & 0 \\ -2/5 & 6/5 & -1 & 0 \\ -1/5 & 3/5 & 0 & -1 \end{bmatrix}$$

The solution to this case is equivalent to case (iii) in [5, Corollary 8].

4. $K = \{2,3\}$.

$$(\hat{\Phi}^K)^{-1} = \begin{bmatrix} 4/5 & 0 & 0 & -6/5 \\ 3/5 & -1 & 0 & 3/5 \\ 1/5 & 0 & -1 & 6/5 \\ -6/5 & 0 & 0 & 24/5 \end{bmatrix}$$

The solution to this case is equivalent to case (iv) in [5, Corollary 8].

5. $K = \{2,3,4\}$. We have:

$$(\hat{\Phi}^K)^{-1} = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 3/4 & -1 & 0 & 0 \\ 1/2 & 0 & -1 & 0 \\ 1/4 & 0 & 0 & -1 \end{bmatrix}$$

and since we always have $1/2\,L_1 - 1/2\,U_1 < 0$, this case is not possible.

It is easily seen that at least one of the four above possible cases generate a proper solution. Therefore, D_A is continuous at 0.

(c) By setting $\bar{\delta}'_4 = \bar{\delta}_3 + \bar{\delta}_4 + \bar{\delta}_5$ and by performing a similar reasoning for $\alpha = 1$ we conclude that D_A is continuous at 1.

(ii) Trivial, because $\alpha \in \{0, 1\}$ generates the trapezoidal approximation. $\square$

Note that the property (ii) alone implies that D_A is bounded. Moreover, we cannot express the lower bound for $D_A(\alpha)$ as a non-trivial function of only $D_A(0)$, because for each $A \in \mathbb{F}^{\pi(\alpha)}(\mathbb{R})$ we get $D_A(\alpha) = 0$.

The continuity of D_A immediately implies the following result.

Theorem 12. *If $A \in \mathbb{F}(\mathbb{R})$ then there exists at least one element $S^* \in \mathbb{F}^{\pi[0,1]}(\mathbb{R})$ such that*

$$\begin{aligned} d(A, S^*) &= \min_{S \in \mathbb{F}^{\pi[0,1]}(\mathbb{R})} d(A, S) \\ &= \min_{\alpha_0 \in [0,1]} d(A, \Pi_{\alpha_0}(A)) \\ &= \min_{\alpha_0 \in [0,1]} D_A(\alpha_0). \end{aligned}$$

5.3. Algorithms

By continuity of D_A we also have that we can always approximate the value $D_A(\alpha)$ with good accuracy, as we can find a convergent process towards the best α_0-piecewise linear approximation on the space $\mathbb{F}^{\pi[0,1]}(\mathbb{R})$. This is because if $(\alpha_n)_{n \geq 1}$ is a sequence of real numbers from the interval $[0, 1]$ and if $\lim_{n\to\infty} \alpha_n = \alpha_0$ then $\lim_{n\to\infty} D_A(\alpha_n) = D_A(\alpha_0)$.

A quite obvious method is to approximate $D_A(\alpha_0)$ by successively dividing the interval $[0, 1]$ in the following way. Firstly we compute $D_A(\alpha_1)$, where $\alpha_1 \in \{0, 1/2, 1\}$ is chosen such that $D_A(\alpha_1)$ is minimal. Then we choose $\alpha_2 \in \{0, 1/4, 1/2, 3/4, 1\}$ such that $D_A(\alpha_2)$ is minimal, and so on. We easily note that $D_A(\alpha_1) \geq D_A(\alpha_2) \geq \dots$.

It can be shown that a convergence rate of this process is quite slow: we have $D_A(\alpha_n) - D_A(\alpha_0) \leq C\,(2^n)^{-1}$, where C is a constant that depends on A, i.e. the convergence rate is linear with respect to the number of points. Although mathematically correct, such approach is of very limited use on a computer. This is because it requires examining 2^n points at the n-th step.

As we do not have any a priori information on the behavior of D_A (except for that it is continuous), in practice we would rather solve this problem via numerical optimization using e.g. a quasi-Newton method (which has at most quadratic rate of convergence) using multiple starting points or some stochastic optimization algorithm.

Example 4. Consider once again the fuzzy number A' from Examples 2 and 3. The $D_{A'}$ function is depicted in Fig. 2. We note that the error for the best piecewise linear approximation is equal to about 20% of that of the nearest trapezoidal approximation. The reduction is therefore significant.

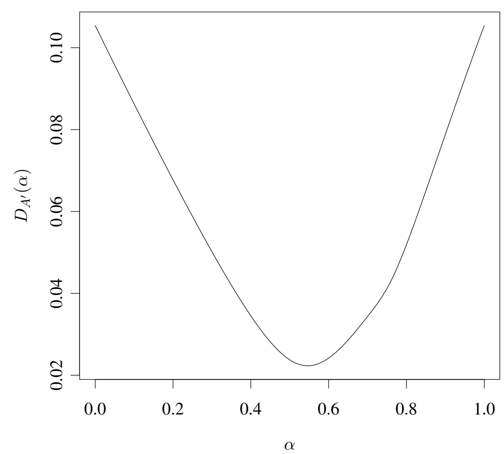

Figure 2: The plot of the $D_{A'}(\alpha) = d(A', \Pi_\alpha(A'))$ function in Example 4.

Here is the result given by the implementation of the algorithm that bases on a convergent sequence.

```
##  N    PTS     ALPHA_N  D(ALPHA_N)
##  1      3 0.500000000 0.023753655
##  2      5 0.500000000 0.023753655
##  3      9 0.500000000 0.023753655
##  4     17 0.562500000 0.022475983
##  5     33 0.531250000 0.022450129
##  6     65 0.546875000 0.022294313
##  7    129 0.546875000 0.022294313
##  8    257 0.546875000 0.022294313
##  9    513 0.546875000 0.022294313
## 10   1025 0.545898438 0.022294097
```

The computation took 6.1 secs. (in the n-th iteration we considered 2^{n-1} new points and compared the result to the previous best solution). We see that the stopping criterion for this algorithm cannot be of the form $D_A(\alpha_n) - D_A(\alpha_{n-1}) < \varepsilon$.

Let us try to solve the problem with a quasi-Newton method (we use the L-BFGS-B algorithm which allows box constraints).

```
optim(0.1, function(alpha) {
   B <- piecewiseLinearApproximation(A, method="NearestEuclidean",
      knot.n=1, knot.alpha=alpha);
   return(distance(A,B));
}, method="L-BFGS-B", lower=1e-9, upper=1-1e-9);

##       PTS     ALPHA_0  D(ALPHA_0)
##        47 0.546235922 0.022294027
```

The computation took 0.8 secs. and required only 47 iterations. We have used $\alpha_1 = 0.1$ as a starting point. We see that we approached a better (in terms of $D_{A'}$) solution than in the above method. ■

Nonconvexity of $\mathbb{F}^{\pi[0,1]}(\mathbb{R})$ (see Remark 6) implies that it might exist $A \in \mathbb{F}(\mathbb{R})$ for which the problem of finding $S^* \in \mathbb{F}^{\pi[0,1]}(\mathbb{R})$ satisfying $d(A, S^*) = \min_{S \in \mathbb{F}^{\pi[0,1]}(\mathbb{R})} d(A, S)$ has not necessarily a unique solution. Indeed, the following example illustrates such case.

Example 5. Let us consider again the fuzzy number A from Example 1. It may be shown (see Fig. 3) that there are two different optimal solutions for the approximation problem of A in this case: $S_1 = \mathrm{S}(1/4, (0, 0, 2/3, 1, 1, 1))$ and $S_2 = \mathrm{S}(3/4, (-1/6, 1/2, 1/2, 1, 1, 1))$. For the 2 minimums at $\alpha_0 = 0.25$ and $\alpha_0' = 0.75$, we have $D_A(\alpha_0) = D_A(\alpha_0') \simeq 0.117851120$. Moreover $D_A(0) = D_A(1) = D_A(0.5)$. Function D_A is given in Fig. 4.

Let us check how the selection of starting point α_1 affects the results obtained. We choose 6 random starting points and get:

```
##     ALPHA_1  PTS     ALPHA_0  D(ALPHA_0)
## 0.076260778   25 0.250094345 0.117851122
## 0.645573585    8 0.749938959 0.117851121
## 0.389381928   25 0.250106880 0.117851122
## 0.500198306   19 0.749826871 0.117851125
## 0.981943340   29 0.749924496 0.117851121
## 0.007671439   28 0.749959380 0.117851121
```

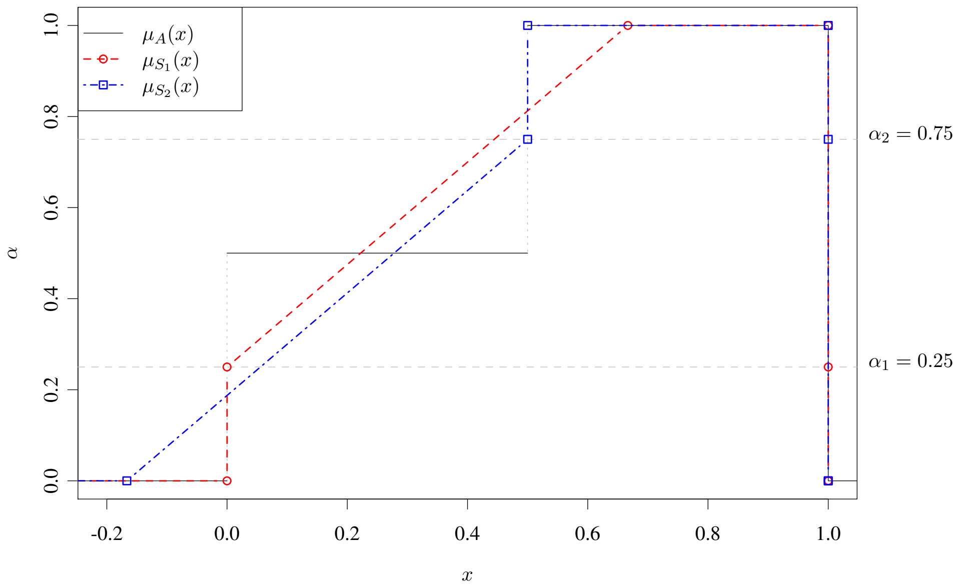

Figure 3: Non-unique solutions S_1, S_2 to the approximation of A in Example 5.

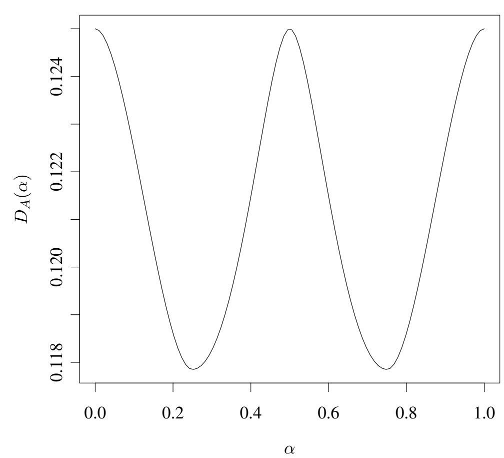

Figure 4: The plot of the $D_A(\alpha) = d(A, \Pi_\alpha(A))$ function in Example 5.

Here and above we used `R`'s default stopping criterion for the L-BFGS-B method, that is convergence occurred when the reduction in the objective function was within 10^{-8}. ■

Unfortunately, it turns out that a local minimum of D_A is not necessarily its global minimum.

Example 6. Consider a fuzzy number A defined with

$$\begin{aligned} A_L(\alpha) &= \begin{cases} 0 & \text{for } \alpha < 0.25, \\ 0.25 & \text{for } 0.25 \le \alpha < 0.5, \\ \alpha & \text{for } 0.5 \le \alpha < 0.75, \\ 1.0 & \text{for } 0.75 \le \alpha, \end{cases} \\ A_U(\alpha) &= 2 - \alpha. \end{aligned}$$

The D_A function has 3 local minima, of which only one is a global minimum, see Fig. 5. This indicates that we need to be very careful while selecting the starting point for the optimization procedure. ■

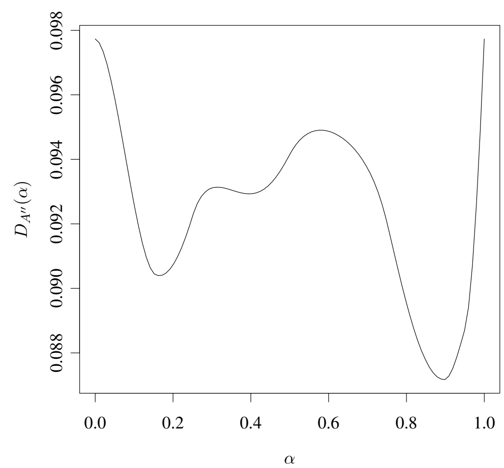

Figure 5: The plot of the $D_{A''}(\alpha) = d(A'', \Pi_\alpha(A''))$ function in Example 6.

6. Conclusion

Using 1-knot fuzzy numbers, one may obtain approximations completely characterized by six points on the real line. They are simple enough and flexible to preserve the main properties of large class of fuzzy quantities.

We have shown that the approximation operator producing piecewise linear 1-knot fuzzy numbers closest to the original fuzzy number possess some desirable properties. Moreover, we have proposed and implemented approximation algorithms that can be applied in practice. Thus in all situations when a trapezoidal approximation is not sufficient we recommend the approximation by 1-knot piecewise linear fuzzy numbers.

Although we have discussed both a case with a fixed knot and the problem of the optimal choice of the knot of the piecewise linear fuzzy number, some problems are, of course, still open. First of all, sometimes a piecewise linear approximation of fuzzy numbers with some additional constraints (on core, support, etc.) would be more adequate. Moreover, one may be interested in approximation of fuzzy numbers by piecewise linear fuzzy numbers having more than one knot. Additionally, the problem of finding the best knot with a more reliable numerical procedure should be examined.

There are also some interesting problems which can be related to the 1-knot piecewise linear approximation. For example, considering the parametric, also known as semi-trapezoidal, approximation of a fuzzy number (see e.g. [5, 27, 36]), it would be interesting to search if for some fuzzy number A we can find the best possible parametric approximation, that is to find $s_L > 0$ and $s_R > 0$ according to the notations in [5]) which provides the best parametric approximation of A. This problem is quite similar to the problem of finding the optimal knot for the best 1-knot piecewise linear approximation. Then it would be interesting to compare both approximations if possible, to see which one is closer to A. Still, for some fuzzy numbers it may happen that the best parametric approximation would not exist since we search the optimal pair (s_L, s_R) in the set $(0, \infty) \times (0, \infty)$ and looking over the definition of the parametric fuzzy numbers it seems that we cannot extend this definition to the case when $s_L = 0$ or $s_R = 0$.

Finally, since in this paper it was pointed out that for some fuzzy numbers the optimal knot for the best piecewise 1-knot approximation may not be unique, it would be of some interest to study whether there are some subsets of fuzzy numbers containing only fuzzy numbers for which the optimal knot would be unique. In this case it would be possible to find other, possibly faster algorithms to determine the optimal knot.

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Appendix A. Proof of Lemma 2

Proof. (i) For simplicity, let $\mathbf{f}_n = (f_{n,1}, f_{n,2}) = \mathrm{S}_e(\alpha_n, \mathbf{c}_n)$ for all $n \geq 1$, where $f_{n,1}$ and $f_{n,2}$ are defined by (7)–(8). By the assumption, $\mathbf{f}_n$ is bounded. Thus, let $M > 0$ be an absolute constant such that $\|\mathbf{f}_n\|_{\widetilde{d}} \leq \sqrt{M}$ for every $n \geq 1$. This implies

$$\|f_{n,1}\|_{[0,1]} \leq \sqrt{M} \text{ and } \|f_{n,2}\|_{[0,1]} \leq \sqrt{M}$$

for all $n \geq 1$. Furthermore, this easily implies that $\|f_{n,1}\|_{[0,\alpha_n]} \leq \sqrt{M}$ and $\|f_{n,2}\|_{[0,\alpha_n]} \leq \sqrt{M}$ for all $n \geq 1$, i.e.

$$\int_0^{\alpha_n} (c_{n,1} + c_{n,2}\,\beta)^2 \, d\beta \leq M, \quad \text{and} \quad \int_0^{\alpha_n} (c_{n,5} + c_{n,6}\,\beta)^2 \, d\beta \leq M$$

for all $n \geq 1$. After elementary calculus we get

$$c_{n,1}^2\,\alpha_n + c_{n,1}\,c_{n,2}\,\alpha_n^2 + c_{n,2}^2\,\alpha_n^3/3 \leq M,$$

and

$$c_{n,5}^2\,\alpha_n + c_{n,5}\,c_{n,6}\,\alpha_n^2 + c_{n,6}^2\,\alpha_n^3/3 \leq M.$$

The first inequality can be written as

$$\alpha_n\left((c_{n,1} + c_{n,2}\,\alpha_n/2)^2 + c_{n,2}^2\,\alpha_n^2/12\right) \leq M,$$

which implies that

$$c_{n,2}^2 \leq 12M/\alpha_n^3 \leq 12M/a^3,$$

because we assumed that for some $a \in (0,1]$ it holds $a \leq \alpha_n$ for all $n \geq 1$. We have thus shown that the sequence $(c_{n,2})_{n\geq 1}$ is bounded. Then we have

$$|c_{n,1} + c_{n,2}\,\alpha_n/2| \leq \sqrt{M/\alpha_n} \leq \sqrt{M/a}$$

which leads to

$$\begin{aligned} |c_{n,1}| &\leq \sqrt{M/a} + |c_{n,2}\,\alpha_n/2| \leq \sqrt{M/a} + |c_{n,2}/2| \\ &\leq \sqrt{M/a} + \sqrt{3M/a^3} \end{aligned}$$

and hence the sequence $(c_{n,1})_{n\geq 1}$ is bounded too. In a similar way we can prove that the sequences $(c_{n,5})_{n\geq 1}$ and $(c_{n,6})_{n\geq 1}$ are also bounded.

(ii) By a similar reasoning we obtain that $\|f_{n,1}\|_{[\alpha_n,1]} \leq \sqrt{M}$, and $\|f_{n,2}\|_{[\alpha_n,1]} \leq \sqrt{M}$ for all $n \geq 1$. This implies:

$$\int_{\alpha_n}^{1} (c_{n,3} + c_{n,4}\,\beta)^2\,d\beta \leq M, \quad \text{and} \quad \int_{\alpha_n}^{1} (c_{n,7} + c_{n,8}\,\beta)^2\,d\beta \leq M$$

for all $n \geq 1$. After calculating the first integral we get:

$$c_{n,3}^2\,(1-\alpha_n) + c_{n,3}\,c_{n,4}\,(1-\alpha_n^2) + c_{n,4}^2\,(1-\alpha_n^3)/3 \leq M.$$

We assumed that $1 - \alpha_n \geq 1 - b$ and therefore:

$$c_{n,3}^2 + c_{n,3}\,c_{n,4}\,(1+\alpha_n) + c_{n,4}^2\,(1+\alpha_n+\alpha_n^2)/3 \leq M/(1-b).$$

The left side of the above inequality may be written as $\left(c_{n,3}+c_{n,4}(1+\alpha_n)/2\right)^2+$ $\left(c_{n,4}\right)^2(1-\alpha_n)^2/12$, and this immediately implies:

$$c_{n,4}^2 \le 12M/\left((1-b)\,(1-\alpha_n)^2\right) \le 12M/(1-b)^3,$$

and, as a consequence, $\left(c_{n,4}\right)_{n\ge 1}$ is bounded.

Then we obtain

$$\begin{aligned} |c_{n,3}| &\le \sqrt{M/(1-b)} + |c_{n,4}\,(1+\alpha_n)/2| \\ &\le \sqrt{M/(1-b)} + |c_{n,4}| \\ &\le \sqrt{M/(1-b)} + \sqrt{12M}/(1-b)^{3/2}, \end{aligned}$$

and therefore $\left(c_{n,3}\right)_{n\ge 1}$ is bounded too. The same estimations for the sequences $\left(c_{n,7}\right)_{n\ge 1}$ and $\left(c_{n,8}\right)_{n\ge 1}$ may easily be obtained.

(iii) Applying the conclusion of assertions (i)-(ii) the proof is immediate and the lemma follows. □

Appendix B. Proof of Lemma 3

Proof. Let $\mathbf{f}_n = (f_{n,1}, f_{n,2}) = \mathrm{S}_e(\alpha_n, \mathbf{c}_n)$ for $n \ge 1$ and $\mathbf{f} = (f_1, f_2) = \mathrm{S}_e(\alpha, \mathbf{c})$. Recall we have

$$\widetilde{d}^2\left(\mathbf{f}_n, \mathbf{f}\right) = \int_0^1 \left(f_{n,1}(\beta) - f_1(\beta)\right)^2 d\beta + \int_0^1 \left(f_{n,2}(\beta) - f_2(\beta)\right)^2 d\beta.$$

We can assume that for all $n \ge 1$ it either holds (i) $\alpha_n \le \alpha$, or (ii) $\alpha_n \ge \alpha$, because the remaining cases can be reduced to these two by using convergent subsequences.

(i) If $\alpha_n \le \alpha$ for all $n \ge 1$, then

$$\begin{aligned} \int_0^1 \left(f_{n,1}(\beta) - f_1(\beta)\right)^2 d\beta &= \int_0^{\alpha_n} \left((c_{n,1} + c_{n,2}\,\beta) - (c_1 + c_2\,\beta)\right)^2 d\beta \\ &+ \int_{\alpha_n}^{\alpha} \left((c_{n,3} + c_{n,4}\,\beta) - (c_1 + c_2\,\beta)\right)^2 d\beta + \int_{\alpha}^{1} \left((c_{n,3} + c_{n,4}\,\beta) - (c_3 + c_4\,\beta)\right)^2 d\beta. \end{aligned}$$

Our hypothesis implies that all the sequences $(c_{n,i})_{n\geq 1}$, $i \in \{1,\ldots,8\}$, are bounded and therefore there exists an absolute constant $M > 0$ such that $|c_{n,i}| \leq M$ for every $n \geq 1$ and $i \in \{1,\ldots,8\}$. As a consequence,

$$0 \leq \int\limits_{\alpha_n}^{\alpha} \big((c_{n,3} + c_{n,4}\,\beta) - (c_1 + c_2\,\beta)\big)^2\, d\beta \leq 16M^2\,(\alpha - \alpha_n) \to 0.$$

Then, we observe that

$$\begin{aligned} 0 \;&\leq\; \int\limits_{0}^{\alpha_n} \big((c_{n,1} + c_{n,2}\,\beta) - (c_1 + c_2\,\beta)\big)^2\, d\beta \\ &\leq\; \int\limits_{0}^{1} \big((c_{n,1} + c_{n,2}\,\beta) - (c_1 + c_2\,\beta)\big)^2\, d\beta \\ &=\; \big\|(c_{n,1} + c_{n,2}\,\alpha) - (c_1 + c_2\,\alpha)\big\|_{[0,1]} \to 0. \end{aligned}$$

Similarly we obtain

$$\int\limits_{\alpha}^{1} \big((c_{n,3} + c_{n,4}\,\beta) - (c_3 + c_4\,\beta)\big)^2\, d\beta \to 0.$$

Therefore, $0 \leq \int_0^1 \big(f_{n,1}(\beta) - f_1(\beta)\big)^2\, d\beta$ is upper bounded by the sum of three sequences, each converging to 0. It implies that $\int_0^1 \big(f_{n,1}(\beta) - f_1(\beta)\big)^2\, d\beta \to 0$. We may obtain $\int_0^1 \big(f_{n,2}(\beta) - f_2(\beta)\big)^2\, d\beta \to 0$ similarly and thus we obtain the desired conclusion.

(ii) The proof for the case $\alpha_n \geq \alpha$ for all $n \geq 1$ is similar. □

Appendix C. Proof of Proposition 4

Proof. Let $(\mathrm{S}_e(\alpha_n, \mathbf{c}_n))_{n\geq 1}$, where $\mathbf{c}_n = (c_{n,1},\ldots,c_{n,8})$ and $\mathrm{S}_e(\alpha_n, \mathbf{c}_n) \in \mathbb{F}^{\pi[0,1]}(\mathbb{R})$, be any convergent sequence in $L^2[0,1] \times L^2[0,1]$. We set $\mathbf{f}_n = (f_{n,1}, f_{n,2}) := \mathrm{S}_e(\alpha_n, \mathbf{c}_n)$ for $n \geq 1$. Moreover, let $\lim^{(\widetilde{d})}_{n\to\infty} \mathbf{f}_n = \mathbf{f} = (f_1, f_2)$.

In order to prove the proposition we have to show that $\mathbf{f} \in \mathbb{F}^{\pi[0,1]}(\mathbb{R})$. However, it suffices to prove that $(\mathbf{f}_n)_{n\geq 1}$ has at least one subsequence which converges to an element of the set $\mathbb{F}^{\pi[0,1]}(\mathbb{R})$ because the uniqueness of the limit implies that $\mathbf{f}$ must be equal to this element and hence we obtain $\mathbf{f} \in \mathbb{F}^{\pi[0,1]}(\mathbb{R})$.

Without loss of generality we may assume that the sequence $(\alpha_n)_{n\geq 1}$ is convergent because otherwise we can choose a convergent subsequence. Therefore, let $\lim_{n\to\infty} \alpha_n = \alpha$. We may distinguish three cases: (i) $\alpha \in (0,1)$, (ii) $\alpha = 0$, and (iii) $\alpha = 1$.

(i) Let $\alpha \in (0,1)$. By case (iii) of Lemma 2 all the sequences $(c_{n,i})_{n\geq 1}$, $i \in \{1,\ldots,8\}$, are bounded. Hence, we may assume that they are convergent (if not then we choose their convergent subsequences with the same indexing and perform identical reasoning). Therefore, for each $i \in \{1,\ldots,8\}$ let $\lim_{n\to\infty} c_{n,i} = c_i$. By Lemma 3 we have $\lim^{(\widetilde{d})}_{n\to\infty} \mathrm{S}_e(\alpha_n, \mathbf{c}_n) = \mathrm{S}_e(\alpha, \mathbf{c}) = \mathbf{f}$, where $\mathbf{c} = (c_1,\ldots,c_8)$ and hence $\mathbf{f} \in \mathbb{F}^{\pi[0,1]}(\mathbb{R})$.

(ii) Let $\alpha = 0$. We may assume that there are only two possibilities here: either $\alpha_n = 0$ for all $n \geq 1$, or $\alpha_n > 0$ for all $n \geq 1$. Indeed, by considering convergent subsequences the remaining cases can be reduced to that two indicated.

If $\alpha_n = 0$ for all $n \geq 1$ then $(\mathbf{f}_n)_{n\geq 1}$ is a convergent sequence of trapezoidal fuzzy numbers. It is well-known (see e.g. [33] or [10, Lemma 3]) that the set of trapezoidal fuzzy numbers is a closed (and even convex) subset of the space $L^2[0,1] \times L^2[0,1]$. Thus $\mathbf{f}$ is a trapezoidal fuzzy number and since $\mathbb{F}^T(\mathbb{R}) \subseteq \mathbb{F}^{\pi[0,1]}(\mathbb{R})$ we get the desired conclusion.

If $\alpha_n > 0$ and $\alpha_n \to 0$ then without loss of generality we may assume that the hypothesis in case (ii) of Lemma 2 is satisfied and therefore the sequences $(c_{n,i})_{n\geq 1}$, $i \in \{3,4,7,8\}$, are all bounded and convergent. Assume $\lim_{n\to\infty} c_{n,i} = c_i$, $i \in \{3,4,7,8\}$. Clearly, $\mathbf{t} = (t_1,t_2)$, such that $t_1(\beta) = c_3 + c_4\,\beta$ and $t_2(\beta) = c_7 + c_8\,\beta$, is a trapezoidal fuzzy number. Now, since $\|\mathbf{f}_n - \mathbf{f}\|_{\widetilde{d}} \to 0$, we have $\|f_{n,1} - f_1\|_{[0,1]} \to 0$ and $\|f_{n,2} - f_2\|_{[0,1]} \to 0$. Thus,

$$\|f_{n,1} - f_1\|_{[a,1]} \to 0, \tag{C.1}$$

and

$$\|f_{n,2} - f_2\|_{[a,1]} \to 0 \tag{C.2}$$

for any arbitrarily chosen $a \in (0,1]$. On the other hand, since $\alpha_n \to 0$, for sufficiently large n, say $n \geq n_0$ we have $\alpha_n < a$. By (7)–(8) it follows that $f_{n,1}(\beta) = c_{n,3} + c_{n,4}\,\beta$ and $f_{n,2}(\beta) = c_{n,7} + c_{n,8}\,\beta$ for every $n \geq n_0$ and $\alpha \in [a,1]$. Then, since $c_{n,i} \to c_i$, $i \in \{3,4,7,8\}$, we get $\|f_{n,1} - t_1\|_{[a,1]} \to 0$ and $\|f_{n,2} - t_2\|_{[a,1]} \to 0$. Therefore, by (C.1)–(C.2) we obtain $f_1 = t_1$ and

$f_2 = t_2$ almost everywhere $\alpha \in [a, 1]$. Since $a \in (0, 1]$ was chosen arbitrarily it follows that $f_1 = t_1$ and $f_2 = t_2$ almost everywhere $\alpha \in [0, 1]$ which implies that we have $\mathbf{f} = \mathbf{t}$ and thus $\mathbf{f} \in \mathbb{F}^{\pi[0,1]}(\mathbb{R})$.

(iii) The proof of the case $\alpha = 1$ is similar to the case (ii). $\square$

Appendix D. Proof of Proposition 5

Proof. Let $(\mathrm{S}_e(\alpha_n, \mathbf{c}_n))_{n \geq 1}$, where $\mathbf{c}_n = (c_{n,1}, \ldots, c_{n,8})$ and $\mathrm{S}_e(\alpha_n, \mathbf{c}_n) \in \mathbb{F}^{\pi[a,b]}(\mathbb{R})$. Moreover, let $\lim^{(\widetilde{d})}_{n\to\infty} \mathbf{f}_n = \mathbf{f} = (f_1, f_2)$. Following step by step the proof of the previous proposition, we observe that $\mathbf{f} \in \mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})$ where there exists a subsequence $(\alpha_{k_n})_{n\geq 1}$ of the sequence $(\alpha_n)_{n\geq 1}$ such that $\alpha_{k_n} \to \alpha_0$. The hypothesis implies that $a \leq \alpha_n \leq b$ and it is easily seen now that $\alpha_0 \in [a, b]$. Therefore, since $\mathbb{F}^{\pi(\alpha_0)}(\mathbb{R}) \subseteq \mathbb{F}^{\pi[a,b]}(\mathbb{R})$ we conclude that $\mathbf{f} \in \mathbb{F}^{\pi[a,b]}(\mathbb{R})$. $\square$